MATH 2010E ADVANCED CALCULUS I
LECTURE 8

WING HONG TONY WONG

14.3 — Partial derivatives

Definition 1. Let $D \subseteq \mathbb{R}^n$ be an open domain. Let $f : D \to \mathbb{R}$ be a real-valued function. Let $\mathbf{a} = (a_1, \ldots, a_n) \in D$. The **partial derivative** of f at $\mathbf{x} = \mathbf{a}$ with respect to the i-th variable x_i is

$$\frac{\partial f}{\partial x_i}(a_1, \ldots, a_n) = f_{x_i}(a_1, \ldots, a_n) = \lim_{h \to 0} \frac{f(a_1, \ldots, a_{i-1}, a_i + h, a_{i+1}, \ldots, a_n) - f(a_1, \ldots, a_n)}{h}.$$

Geometrically, when we consider the partial derivative $\frac{\partial f}{\partial x_i}(a_1, \ldots, a_n)$, we first consider the curve obtained by intersecting the surface

$$x_{n+1} = f(x_1, \ldots, x_n)$$

with the 2-dimensional plane

$$x_1 = a_1,\ \ldots,\ x_{i-1} = a_{i-1},\ x_{i+1} = a_{i+1},\ \ldots,\ x_n = a_n.$$

The partial derivative is the slope of the curve at the point $x_i = a_i$ on the 2-dimensional plane. (*Please refer to Dr. Martin Li's* 2010*B Lecture Note Week* 6 *P.*4 *for the picture.*)

By comparing this definition of partial derivative with that of the derivative of single-valued functions, we see that we are essentially taking the derivative of $f(x_1, \ldots, x_n)$ with respect to x_i by treating other variables as constants.

Example 2. Let $f(x, y) = \begin{cases} \dfrac{\sin xy}{xy} & \text{if } xy \neq 0 \\ 1 & \text{otherwise} \end{cases}$. Find $f_x(2, 1)$, $f_y(0, 1)$, and $f_x(0, 1)$.

Solution. When $xy \neq 0$,

$$f_x(x, y) = \frac{\partial}{\partial x} \frac{\sin xy}{xy} = -\frac{\sin xy}{x^2 y} + \frac{y \cos xy}{xy}.$$

So $f_x(2, 1) = -\dfrac{\sin 2}{4} + \dfrac{\cos 2}{2} = \dfrac{2\cos 2 - \sin 2}{4}$.

$$f_y(0, 1) = \lim_{h \to 0} \frac{f(0, 1 + h) - f(0, 1)}{h} = \lim_{h \to 0} \frac{1 - 1}{h} = 0.$$

$$f_x(0, 1) = \lim_{h \to 0} \frac{f(h, 1) - f(0, 1)}{h} = \lim_{h \to 0} \frac{\frac{\sin h}{h} - 1}{h} = \lim_{h \to 0} \frac{\sin h - h}{h^2}.$$

Since $\lim\limits_{h \to 0} \sin h - h = \lim\limits_{h \to 0} h^2 = 0$, we consider $\lim\limits_{h \to 0} \dfrac{(\sin h - h)'}{(h^2)'} = \lim\limits_{h \to 0} \dfrac{\cos h - 1}{2h}$.

Since $\lim\limits_{h \to 0} \cos h - 1 = \lim\limits_{h \to 0} 2h = 0$, we consider $\lim\limits_{h \to 0} \dfrac{(\cos h - 1)'}{(2h)'} = \lim\limits_{h \to 0} \dfrac{-\sin h}{2} = 0$.

By l'Hôpital's rule, $f_x(0, 1) = 0$. □

Date: Thursday, 4th June, 2015.

Example 3. Let $x^2 + y^2 \ln z + z^2 \sin x = xyz$ be defined in a domain of $\mathbb{R}^3$ such that z is a function of x and y. Find $\dfrac{\partial z}{\partial x}\left(\dfrac{\pi}{2}, 2, 1\right)$.

To do this type of problem, we treat all independent variables except x as constants.

Solution.

$$\begin{aligned}
\frac{\partial}{\partial x}(x^2 + y^2 \ln z + z^2 \sin x) &= \frac{\partial}{\partial x} xyz \\
2x + y^2 \frac{1}{z}\frac{\partial z}{\partial x} + 2z \sin x \frac{\partial z}{\partial x} + z^2 \cos x &= yz + xy\frac{\partial z}{\partial x} \\
\frac{\partial z}{\partial x}\left(\frac{y^2}{z} + 2z\sin x - xy\right) &= yz - 2x - z^2 \cos x \\
\frac{\partial z}{\partial x} &= \frac{yz - 2x - z^2\cos x}{\frac{y^2}{z} + 2z \sin x - xy} \\
\frac{\partial z}{\partial x}\left(\frac{\pi}{2}, 2, 1\right) &= \frac{2 - \pi - 1\cos\frac{\pi}{2}}{4 + 2\sin\frac{\pi}{2} - \pi} = \frac{2-\pi}{6-\pi}
\end{aligned}$$

$\square$

If we want to take **higher order partial derivatives**, we need to take first order partial derivatives one-by-one repeatedly. For example,

$$f_{zy} = \frac{\partial^2 f}{\partial y \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right),\ f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right),$$
$$f_{yyz} = \frac{\partial^3 f}{\partial z \partial y^2} = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\right),\ \text{and } f_{yxy} = \frac{\partial^3 f}{\partial y \partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\right) \text{ etc.}$$

Example 4. Let $f(x,y) = \begin{cases} x^2 \sin\left(\dfrac{y}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{otherwise} \end{cases}$. Find $f_{xy}(1,1)$, $f_{yx}(1,1)$, $f_{xy}(0,0)$, and $f_{yx}(0,0)$.

Solution. When $x \neq 0$,

$$f_x = 2x\sin\left(\frac{y}{x}\right) + x^2\cos\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) = 2x\sin\left(\frac{y}{x}\right) - y\cos\left(\frac{y}{x}\right),$$

and

$$f_y = x^2\cos\left(\frac{y}{x}\right)\cdot\frac{1}{x} = x\cos\left(\frac{y}{x}\right).$$

Hence,

$$\begin{aligned}
f_{xy} &= \frac{\partial}{\partial y}\left[2x\sin\left(\frac{y}{x}\right) - y\cos\left(\frac{y}{x}\right)\right] \\
&= 2x\cos\left(\frac{y}{x}\right)\cdot\frac{1}{x} - \cos\left(\frac{y}{x}\right) + y\sin\left(\frac{y}{x}\right)\cdot\frac{1}{x} \\
&= \cos\left(\frac{y}{x}\right) + \frac{y}{x}\sin\left(\frac{y}{x}\right),
\end{aligned}$$

and

$$\begin{aligned}
f_{yx} &= \frac{\partial}{\partial x}\left[x\cos\left(\frac{y}{x}\right)\right] \\
&= \cos\left(\frac{y}{x}\right) - x\sin\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) \\
&= \cos\left(\frac{y}{x}\right) + \frac{y}{x}\sin\left(\frac{y}{x}\right).
\end{aligned}$$

Therefore, $f_{xy}(1,1) = f_{yx}(1,1) = \cos 1 + \sin 1$.

When $x = 0$,

$$f_x(0,y) = \lim_{h\to 0}\frac{f(0+h,y) - f(0,y)}{h} = \lim_{h\to 0}\frac{h^2\sin\left(\frac{y}{h}\right) - 0}{h} = \lim_{h\to 0} h\sin\left(\frac{y}{h}\right) = 0$$

by sandwich theorem. This implies

$$f_{xy}(0,0) = \left.\frac{\partial}{\partial y}\right|_{(0,0)} f_x = \lim_{h\to 0}\frac{f_x(0,h) - f_x(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0.$$

On the other hand,

$$f_y(0,y) = \lim_{h\to 0}\frac{f(0,y+h) - f(0,y)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0.$$

This implies

$$\begin{aligned}
f_{yx}(0,0) = \left.\frac{\partial}{\partial x}\right|_{(0,0)} f_y &= \lim_{h\to 0}\frac{f_y(h,0) - f_y(0,0)}{h} \\
&= \lim_{h\to 0}\frac{h\cos\left(\frac{0}{h}\right) - 0}{h} = \lim_{h\to 0}\cos 0 = 1.
\end{aligned}$$

□

Warning: From this example, note that $f_{xy} \neq f_{yx}$ in general.

Theorem 5 (Mixed derivative theorem)**.** *Let D be an open domain. Let $f : D \to \mathbb{R}$ be a real-valued function. Let $\mathbf{a} \in D$. If f, f_x, f_y, f_{xy}, f_{yx} are all defined in D and are all continuous at $\mathbf{a}$ (i.e. f is C^2 at $\mathbf{a}$), then*

$$f_{xy}(\mathbf{a}) = f_{yx}(\mathbf{a}).$$

Example 6. Let $w = xy + \dfrac{e^y}{y^2+1}$. Find $\dfrac{\partial^2 w}{\partial x \partial y}$.

Solution. Method 1.

$$\frac{\partial^2 w}{\partial x\partial y} = \frac{\partial}{\partial x}\left(x - \frac{2ye^y}{(y^2+1)^2} + \frac{e^y}{y^2+1}\right) = 1.$$

Method 2. Since the function w is a composition of perfectly smooth function and the denominator is nonzero, w is obviously a C^2 function. By mixed derivative theorem,

$$\frac{\partial^2 w}{\partial x\partial y} = \frac{\partial^2 w}{\partial y\partial x} = \frac{\partial}{\partial y}y = 1.$$

□

Proof of mixed derivative theorem. Before we proceed, let us recall the mean value theorem for one variable functions.

Theorem 7 (Mean value theorem)**.** *Let $\chi : [m,n] \to \mathbb{R}$ be a differentiable function. Then there exists a real number $c \in (m,n)$ such that*

$$\chi'(c) = \frac{\chi(m) - \chi(n)}{m - n},$$

or

$$\chi(m) - \chi(n) = \chi'(c)(m - n).$$

In this proof of mixed derivative theorem, we will assume $D \in \mathbb{R}^2$. Let $\mathbf{a} = (a,b)$. Consider the quantity

$$\Delta = f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a,b).$$

$$\Delta = \big[f(a+h,b+k) - f(a+h,b)\big] - \big[f(a,b+k) - f(a,b)\big].$$

Let $\Phi(x) = f(x, b+k) - f(x,b)$.
Then $\Delta = \Phi(a+h) - \Phi(a)$.
By mean value theorem, $\Delta = \Phi'(a+h_1)h$, where $h_1 \in (0,h)$.
By definition of Φ, we have
$\Delta = \big[f_x(a+h_1, b+k) - f_x(a+h_1,b)\big]h$.
Next, let $\phi(y) = f_x(a+h_1, y)$.
Then $\Delta = \big[\phi(b+k) - \phi(b)\big]h$.
By mean value theorem, $\Delta = \phi'(b+k_1)kh$, where $k_1 \in (0,k)$.
By definition of ϕ, we have
$\Delta = f_{xy}(a+h_1, b+k_1)kh$.

$$\Delta = \big[f(a+h,b+k) - f(a,b+k)\big] - \big[f(a+h,b) - f(a,b)\big].$$

Let $\Psi(y) = f(a+h, y) - f(a,y)$.
Then $\Delta = \Psi(b+k) - \Psi(b)$.
By mean value theorem, $\Delta = \Psi'(b+k_2)k$, where $k_2 \in (0,k)$.
By definition of Ψ, we have
$\Delta = \big[f_y(a+h, b+k_2) - f_y(a, b+k_2)\big]k$.
Next, let $\psi(x) = f_y(x, b+k_2)$.
Then $\Delta = \big(\psi(a+h) - \psi(a)\big)k$.
By mean value theorem, $\Delta = \psi'(a+h_2)hk$, where $h_2 \in (0,h)$.
By definition of ψ, we have
$\Delta = f_{yx}(a+h_2, b+k_2)hk$.

Hence,

$$f_{xy}(a+h_1, b+k_1) = f_{yx}(a+h_2, b+k_2).$$

Since f_{xy} and f_{yx} are continuous in at (a,b),

$$\begin{aligned}
\lim_{h\to 0}\lim_{k\to 0} f_{xy}(a+h_1, b+k_1) &= \lim_{h\to 0}\lim_{k\to 0} f_{yx}(a+h_2, b+k_2) \\
f_{xy}\left(a + \lim_{h\to 0} h_1, b + \lim_{k\to 0} k_1\right) &= f_{yx}\left(a + \lim_{h\to 0} h_2, b + \lim_{k\to 0} k_2\right) \\
f_{xy}(a,b) &= f_{yx}(a,b).
\end{aligned}$$

□

Recall that the i-th partial derivative of f at $\mathbf{x} = \mathbf{a}$ is the derivative (slope) of the curve at $x_i = a_i$ obtained by intersecting the surface

$$x_{n+1} = f(x_1, \ldots, x_n)$$

and the 2-dimensional plane

$$x_1 = a_1,\ \ldots,\ x_{i-1} = a_{i-1},\ x_{i+1} = a_{i+1},\ \ldots,\ x_n = a_n.$$

Hence, the existence of the i-th derivative of f for all $i = 1, 2, \ldots, n$ does not guarantee "differentiability" of f. In fact, we have the following warning for you.

Warning: The existence of f_x and f_y does not even guarantee the continuity of f.

Example 8. Let $f(x,y) = \begin{cases} 0 & \text{if } xy = 0 \\ 1 & \text{otherwise} \end{cases}$.
(a) Find $f_x(0,0)$ and $f_y(0,0)$.
(b) Is this function f continuous at $(0,0)$?

Solution. (a)

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

(b) $f(0,0) = 0$. If f is continuous, then $\lim_{(x,y)\to(0,0)} f(x,y) = 0$. Note that

$$\lim_{\substack{(x,y)=(t,t)\\ t\to 0}} f(x,y) = \lim_{t\to 0} f(t,t) = \lim_{t\to 0} 1 = 1.$$

Therefore, f is not continuous at $(0,0)$.

□

Here, we will state without proof the following criteria of "differentiability" of f.

Theorem 9. *Let D be an open domain. If all the i-th partial derivatives of f are continuous at every point in D (i.e. f is C^1 at every point in D), then f is "differentiable" and hence continuous at every point in D.*

However, what is the definition of "differentiability"?

Definition 10. Let $D \subseteq \mathbb{R}^n$ be an open domain. Let $f : D \to \mathbb{R}$ be a real-valued function. Let $\mathbf{a} = (a_1, \ldots, a_n) \in D$. Then f is **differentiable at $\mathbf{x} = \mathbf{a}$** if

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - \nabla f(\mathbf{a})\cdot\mathbf{h}}{\|\mathbf{h}\|} = 0,$$

where $\nabla f(\mathbf{a}) = \big(f_{x_1}(\mathbf{a}), f_{x_2}(\mathbf{a}), \ldots, f_{x_n}(\mathbf{a})\big)$, called the **gradient vector** of f at $\mathbf{x} = \mathbf{a}$. If f is differentiable at every point in D, then f is **differentiable**.

Recall that in one-dimensional case,

$$\begin{aligned} \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} &= f'(a) \\ \lim_{h\to 0} \frac{f(a+h) - f(a) - f'(a)\cdot h}{h} &= 0 \\ \lim_{h\to 0} \frac{f(a+h) - f(a) - f'(a)\cdot h}{|h|} &= 0, \end{aligned}$$

so Definition 10 is a generalization of the one-dimensional case.

Remark: In two-dimensional case, f is differentiable at $(x,y) = (a,b)$ if

$$\lim_{(h,k)\to(0,0)} \frac{f(a+h,b+k) - f(a,b) - f_x(a,b)h - f_y(a,b)k}{\sqrt{h^2+k^2}} = 0.$$

Although we have motivated from one-dimensional case, the generalization is still not obvious. If we revisit the definition of "differentiable" in one-dimensional case, we have f is differentiable at $x = a$ if

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists.

If we directly generalize this definition to higher-dimension, we have f is differentiable at $\mathbf{x} = \mathbf{a}$ if

$$\text{``}\lim_{\mathbf{h} \to \mathbf{0}} \frac{f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a})}{\mathbf{h}}\text{''}$$

exists. Unfortunately, the denominator is a vector, which does not make sense. If we use the limit

$$\text{``}\lim_{\mathbf{h} \to \mathbf{0}} \frac{f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a})}{\|\mathbf{h}\|}\text{''}$$

instead, then if $f(\mathbf{x}) = x_1$, the limit yields

$$\lim_{\mathbf{h} \to \mathbf{0}} \frac{(a_1 + h_1) - a_1}{\|\mathbf{h}\|} = \lim_{\mathbf{h} \to \mathbf{0}} \frac{h_1}{\|\mathbf{h}\|}$$

which does not exist (take two paths: $\mathbf{h} = t(1, 0, \ldots, 0)$ and $\mathbf{h} = t(0, 1, 0 \ldots, 0)$, $t \to 0$). This is unacceptable since we want $f(\mathbf{x}) = x_1$ to be differentiable.

In order to understand Definition 10 better, we need to think about the geometric interpretation of differentiability and derivatives, i.e. tangent line and tangent plane.

In one-dimensional case, the **tangent vector** of $y = f(x)$ at the point $x = a$ is $\big(1, f'(a)\big)$, so the parametric form of the **tangent line** is

$$\left\{ \big(a, f(a)\big) + t\big(1, f'(a)\big) : t \in \mathbb{R} \right\}.$$

Hence, $x = a + t$, and $y = f(a) + tf'(a)$. By substituting $t = x - a$ into the second equation, we get

$$y = f(a) + f'(a)(x - a).$$

In higher-dimensional case, the **tangent vectors** of $x_{n+1} = f(\mathbf{x})$ at the point $\mathbf{x} = \mathbf{a}$ are $\big(1, 0, \ldots, 0, f_{x_1}(\mathbf{a})\big)$, $\ldots$, $\big(0, \ldots, 0, 1, f_{x_n}(\mathbf{a})\big)$, so the parametric form of the **tangent plane** is

$$\left\{ \big(a_1, \ldots, a_n, f(\mathbf{a})\big) + t_1\big(1, 0, \ldots, 0, f_{x_1}(\mathbf{a})\big) + \cdots + t_n\big(0, \ldots, 0, 1, f_{x_n}(\mathbf{a})\big) : t_1, \ldots, t_n \in \mathbb{R} \right\}.$$

(*Please refer to Dr. Martin Li's* 2010*B Lecture Note Week* 7 *P.*6 *for the picture.*)
Hence, $x_i = a_i + tf_{x_i}(\mathbf{a})$ for all $i = 1, \ldots, n$, and $x_{n+1} = f(\mathbf{a}) + t_1 f_{x_1}(\mathbf{a}) + \cdots + t_n f_{x_n}(\mathbf{a})$. By substituting $t_i = x_i - a_i$ into the last equation, we get

$$\begin{aligned} x_{n+1} &= f(\mathbf{a}) + f_{x_1}(\mathbf{a})(x_1 - a_1) + \cdots + f_{x_n}(\mathbf{a})(x_n - a_n) \\ &= f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (x_1 - a_1, \ldots, x_n - a_n). \end{aligned}$$

Remark: In two-dimensional case, the equation of the **tangent plane** of $z = f(x, y)$ at the point $(x, y) = (a, b)$ is

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

For f to be differentiable, it means that the **tangent plane approximation** of $f(\mathbf{x})$ around $\mathbf{x} = \mathbf{a}$ converges faster than $\mathbf{x} \to \mathbf{a}$. In limit notation,

$$\lim_{\mathbf{x} \to \mathbf{a}} \frac{f(\mathbf{x}) - \big[f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (x_1 - a_1, \ldots, x_n - a_n)\big]}{\|\mathbf{x} - \mathbf{a}\|} = 0,$$

or

$$\lim_{\mathbf{a} \to \mathbf{h}} \frac{f(\mathbf{x} + \mathbf{h}) - f(\mathbf{a}) - \nabla f(\mathbf{a}) \cdot \mathbf{h}}{\|\mathbf{h}\|} = 0.$$