MATH 2010E ADVANCED CALCULUS I
LECTURE 6

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14.2 — Limits and continuity in higher dimensions

Definition 1. Let $D \subseteq \mathbb{R}^n$ be a domain. Let $f : D \to \mathbb{R}$ be a real-valued function. Let $\mathbf{x}_0 \in \overline{D}$ and $L \in \mathbb{R}$. Then

$$\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = L$$

if for every $\epsilon > 0$, there exists $\delta > 0$ such that one of the following occurs.

- for all $\mathbf{x} \in D$ satisfying $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$, $|f(\mathbf{x}) - L| < \epsilon$.
- $f\big((B_\delta(\mathbf{x}_0)\backslash\{\mathbf{x}_0\}) \cap D\big) \subseteq B_\epsilon(L)$.

Example 2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(x, y) = x$. Show that $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y) = x_0$.

Solution. For all $\epsilon > 0$, let $\delta = \epsilon$. For all $(x, y) \in \mathbb{R}^2$ satisfying $0 < \|(x, y) - (x_0, y_0)\| < \delta$, we have

$$\begin{aligned}
|f(x, y) - x_0| &= |x - x_0| \\
&= \sqrt{(x - x_0)^2} \\
&\leq \sqrt{(x - x_0)^2 + (y - y_0)^2} \\
&= \|(x, y) - (x_0, y_0)\| < \delta = \epsilon.
\end{aligned}$$

□

Example 3. Show that $\lim\limits_{(x,y)\to(0,0)} \left(x^2 + y^2\right) = 0$, where the domain is $\mathbb{R}^2$.

Solution. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be such that $f(x, y) = x^2 + y^2$. For all $\epsilon > 0$, let $\delta = \sqrt{\epsilon}$. For all $(x, y) \in \mathbb{R}^2$ satisfying $0 < \|(x, y) - (0, 0)\| < \delta$, we have

$$\begin{aligned}
|f(x, y) - 0| &= |x^2 + y^2| \\
&= \|(x, y)\|^2 < \delta^2 = \epsilon.
\end{aligned}$$

□

If $n = 1$, then $B_\delta(x_0)$ is simply an open interval centered at x_0, and $x \to x_0$ can be split into $x \to x_0^-$ and $x \to x_0^+$. More precisely, we have

$$\lim_{x\to x_0} f(x) = L \text{ if and only if } \lim_{x\to x_0^-} f(x) = L \text{ and } \lim_{x\to x_0^+} f(x) = L.$$

However, when $n = 2$ or higher, there are many ways for $\mathbf{x} \to \mathbf{x}_0$. Even if

$$\lim_{h\to 0} f(\mathbf{x}_0 + h\mathbf{v}) = L$$

for all $\mathbf{v}$ such that $\|\mathbf{v}\| = 1$, the limit may still NOT exist. In fact, $\lim\limits_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = L$ if and only if no matter which path we choose for $\mathbf{x} \to \mathbf{x}_0$, the limit is still L.

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Example 4. Does $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ exist, where the domain is $\mathbb{R}^2\backslash\{(0,0)\}$?

Solution. Note that

$$\lim_{x\to 0}\lim_{y\to 0}\frac{x^2-y^2}{x^2+y^2} = \lim_{x\to 0}\frac{x^2}{x^2} = \lim_{x\to 0} 1 = 1,$$

while

$$\lim_{y\to 0}\lim_{x\to 0}\frac{x^2-y^2}{x^2+y^2} = \lim_{y\to 0}\frac{-y^2}{y^2} = \lim_{y\to 0} -1 = -1.$$

Hence, when $(x,y)\to(0,0)$ using two different paths (the first path is along the x-axis, and the second path is along the y-axis), the limits are different. Therefore, the desired limit does not exist.

□

Example 5. Does $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ exist, where the domain is $\mathbb{R}^2\backslash\{(0,0)\}$?

Solution. Before we proceed, we first observe that

$$\lim_{x\to 0}\lim_{y\to 0}\frac{xy}{x^2+y^2} = \lim_{x\to 0}\frac{0}{x^2} = \lim_{x\to 0} 0 = 0,$$

and

$$\lim_{y\to 0}\lim_{x\to 0}\frac{xy}{x^2+y^2} = \lim_{y\to 0}\frac{0}{y^2} = \lim_{y\to 0} 0 = 0.$$

However, this does NOT imply that the desired limit exist, since we have only checked two paths $(x,y)\to(0,0)$, but we have not checked many other paths.

To finish this problem, we take the paths $(x,y)=(t,kt)$, $t\to 0$, $k\neq 0$.

$$\lim_{\substack{(x,y)=(t,kt)\\ t\to 0}}\frac{xy}{x^2+y^2} = \lim_{t\to 0}\frac{kt^2}{t^2+k^2t^2} = \lim_{t\to 0}\frac{k}{1+k^2} = \frac{k}{1+k^2}.$$

If $k=1$, the limit is $\frac{1}{2}$. Since the limits are different when $(x,y)\to(0,0)$ using different paths, the desired limit does not exist.

□

Example 6. Does $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$ exist, where the domain is $\mathbb{R}^2\backslash\{(0,0)\}$?

Solution. Before we proceed, we first observe that

$$\lim_{x\to 0}\lim_{y\to 0}\frac{x^2y}{x^4+y^2} = \lim_{x\to 0}\frac{0}{x^4} = \lim_{x\to 0} 0 = 0,$$

$$\lim_{y\to 0}\lim_{x\to 0}\frac{x^2y}{x^4+y^2} = \lim_{y\to 0}\frac{0}{y^2} = \lim_{y\to 0} 0 = 0,$$

and

$$\lim_{\substack{(x,y)=(t,kt)\\ t\to 0}}\frac{x^2y}{x^4+y^2} = \lim_{t\to 0}\frac{kt^3}{t^4+k^2t^2} = \lim_{t\to 0}\frac{kt}{t^2+k^2} = 0$$

since $\lim_{t\to 0} kt = 0$ and $\lim_{t\to 0} t^2+k^2 = k^2$. However, these do NOT imply that the desired limit exist, since we have only checked straight-line paths $(x,y)\to(0,0)$, but we have not checked many other paths.

To finish this problem, we take the path $(x,y) = (t,t^2)$, $t \to 0$.

$$\lim_{\substack{(x,y)=(t,t^2)\\ t\to 0}} \frac{x^2y}{x^4+y^2} = \lim_{t\to 0} \frac{t^4}{t^4+t^4} = \lim_{t\to 0} \frac{1}{2} = \frac{1}{2}.$$

Since the limits are different when $(x,y) \to (0,0)$ using different paths, the desired limit does not exist.

$\square$

Theorem 7. *Let $D \subseteq \mathbb{R}^n$ be a domain. Let $f : D \to \mathbb{R}$ be a real-valued function. Let $\mathbf{x}_0 \in \overline{D}$ and $L, M, k \in \mathbb{R}$. If $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = L$ and $\lim_{\mathbf{x}\to\mathbf{x}_0} g(\mathbf{x}) = M$, then*

- *(sum and difference rule)* $\lim_{\mathbf{x}\to\mathbf{x}_0} \big(f(\mathbf{x}) \pm g(\mathbf{x})\big) = L \pm M$.
- *(constant multiple rule)* $\lim_{\mathbf{x}\to\mathbf{x}_0} kf(\mathbf{x}) = kL$.
- *(product rule)* $\lim_{\mathbf{x}\to\mathbf{x}_0} \big(f(\mathbf{x})g(\mathbf{x})\big) = LM$.
- *(quotient rule)* $\lim_{\mathbf{x}\to\mathbf{x}_0} \dfrac{f(\mathbf{x})}{g(\mathbf{x})} = \dfrac{L}{M}$ *if* $M \neq 0$.
- *(power rule)* $\lim_{\mathbf{x}\to\mathbf{x}_0} \big(f(\mathbf{x})\big)^k = L^k$ *if* $L > 0$*; if* $L = 0$*, then we need* $k > 0$*; if* $L < 0$*, then we need* $k = \dfrac{p}{q}$*, where* q *is odd.*

Together with $\lim_{(x,y)\to(x_0,y_0)} x = x_0$ and $\lim_{(x,y)\to(x_0,y_0)} y = y_0$, it can be seen that if f is a function that is a fraction with both numerator and denominator being polynomials or functions involving powers and roots etc, then the limit of f can be evaluated by substituting (x_0,y_0) into (x,y), provided that the denominator is nonzero.

Example 8. Find $\lim_{(x,y)\to(0,0)} \dfrac{x^2-xy}{\sqrt{x}-\sqrt{y}}$, where the domain is $\{(x,y) \in \mathbb{R}^2 : x > 0, y > 0, x \neq y\}$.

Solution. Note that $\lim_{(x,y)\to(0,0)} \left(\sqrt{x}-\sqrt{y}\right) = 0$, so we cannot evaluate the limit by substitution.

$$\begin{aligned} \lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}} &= \lim_{(x,y)\to(0,0)} \frac{x(x-y)}{\sqrt{x}-\sqrt{y}} \\ &= \lim_{(x,y)\to(0,0)} x\left(\sqrt{x}+\sqrt{y}\right) = 0 \end{aligned}$$

by substitution.

$\square$

Example 9. Find $\lim_{(x,y)\to(0,0)} \dfrac{4xy^2}{x^2+y^2}$, where the domain is $\mathbb{R}^2\backslash\{(0,0)\}$.

Solution. Note that $\lim_{(x,y)\to(0,0)} \left(x^2+y^2\right) = 0$, so we cannot evaluate the limit by substitution.

A possible candidate for the limit is 0. For all $\epsilon > 0$, let $\delta = \dfrac{\epsilon}{2}$. For all $(x,y) \in \mathbb{R}^2\backslash\{(0,0)\}$ satisfying $0 < \|(x,y)-(0,0)\| = \sqrt{x^2+y^2} < \delta$, we have

$$\left|\frac{4xy^2}{x^2+y^2} - 0\right| \leq \left|\frac{4xy^2}{2xy}\right| = |2y| = 2\sqrt{y^2} \leq 2\sqrt{x^2+y^2} < 2\delta = \epsilon.$$

Here, the first inequality is due to the AM-GM inequality: $\dfrac{a+b}{2} \geq \sqrt{ab}$ for all $a, b \geq 0$.

$\square$

Theorem 10 (Sandwich theorem)**.** *Let $D \subseteq \mathbb{R}^n$ be a domain. Let $f, g, h : D \to \mathbb{R}$ be real-valued functions. Let $\mathbf{x}_0 \in \overline{D}$ and $L \in \mathbb{R}$. If*

- *$h(\mathbf{x}) \leq f(\mathbf{x}) \leq g(\mathbf{x})$ for all $\mathbf{x} \in D$, and*
- *$\lim_{\mathbf{x}\to\mathbf{x}_0} g(\mathbf{x}) = \lim_{\mathbf{x}\to\mathbf{x}_0} h(\mathbf{x}) = L$,*

then $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = L$.

In particular, if

- *$|f(\mathbf{x})| \leq g(\mathbf{x})$ for all $\mathbf{x} \in D$, and*
- *$\lim_{\mathbf{x}\to\mathbf{x}_0} g(\mathbf{x}) = 0$,*

then $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = 0$.

Example 11. Find $\lim_{(x,y)\to(0,0)} x\cos\left(\dfrac{1}{x^2+y^2}\right)$, where the domain is $\mathbb{R}^2\backslash\{(0,0)\}$.

Solution. Note that $\left|x\cos\left(\dfrac{1}{x^2+y^2}\right)\right| \leq |x|$ for all (x, y) in the domain, and $\lim_{(x,y)\to(0,0)} |x| = 0$. By sandwich theorem, $\lim_{(x,y)\to(0,0)} x\cos\left(\dfrac{1}{x^2+y^2}\right) = 0$.

$\square$

Continuity

Definition 12. Let $D \subseteq \mathbb{R}^n$ be a domain. Let $f : D \to \mathbb{R}$ be a real-valued function. Let $\mathbf{x}_0 \in D$. Then f is **continuous** at $\mathbf{x}_0$ if

$$\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = f(\mathbf{x}_0).$$

If f is continuous at every $\mathbf{x}_0 \in D$, then f is **continuous**.

Example 13. Show that the function

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{otherwise} \end{cases}$$

is continuous at every point $(x_0, y_0) \in \mathbb{R}^2$ except $(0, 0)$.

Solution. At $(x_0, y_0) \neq (0, 0)$, $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ can be evaluated by substituting (x_0, y_0) into (x, y). In other words, $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$.

However, at $(x_0, y_0) = (0, 0)$, the limit was shown that it does not exist in Example 5. Therefore, f is discontinuous at $(0, 0)$.

$\square$

Example 14. Show that the function

$$f(x,y) = \begin{cases} \dfrac{4xy^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{otherwise} \end{cases}$$

is continuous at every point $(x_0, y_0) \in \mathbb{R}^2$.

Solution. At $(x_0, y_0) \neq (0,0)$, $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ can be evaluated by substituting (x_0, y_0) into (x, y). In other words, $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$.

In Example 9, it was shown that $\lim\limits_{(x,y)\to(0,0)} \dfrac{4xy^2}{x^2+y^2} = 0 = f(0,0)$. Therefore, f is continuous at $(0,0)$ as well.

□

Theorem 15. *Let $D \subseteq \mathbb{R}^n$ be a domain. Let $f : D \to \mathbb{R}$ be a real-valued function. Let $\mathbf{x}_0 \in D$. Let $E \subseteq \mathbb{R}$ be such that $f(\mathbf{x}_0) \in E$. Let $g : E \to \mathbb{R}$ be a scalar function. If f is continuous at $\mathbf{x}_0$ and g is continuous at $f(\mathbf{x}_0)$, then $g \circ f$ is continuous at $\mathbf{x}_0$.*

Example 16. Show that

$$h(x,y) = \begin{cases} \sqrt[3]{\dfrac{4xy^2}{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{otherwise} \end{cases}$$

is continuous $(0,0)$.

Solution. Let $f(x,y)$ be the function in Example 14, and let $g(x) = \sqrt[3]{x}$. Note that $h(x,y) = g\big(f(x,y)\big)$. Since f is continuous at $(0,0)$, and g is continuous at $f(0,0) = 0$, by Theorem 15, h is continuous at $(0,0)$.

□

Polar coordinates in $\mathbb{R}^2$

Every point (x, y) in $\mathbb{R}^2$ can be represented by (r, θ), where $r \geq 0$, and $\theta \in [0, 2\pi)$. It is defined such that $x = r\cos\theta$, and $y = r\sin\theta$. In other words, $r = \sqrt{x^2+y^2}$, and $\theta = \tan^{-1}\dfrac{y}{x} = \sin^{-1}\dfrac{y}{r} = \cos^{-1}\dfrac{x}{r}$. Note that θ is not well-defined when $r = 0$.

Polar coordinates can sometimes simplify many clumsy expressions in rectangular coordinates. For example, the equation of the circle in rectangular coordinates is $x^2+y^2 = R^2$, while the equation of the same circle in polar coordinates is $r = R$.

Example 17 (Archemedean spiral)**.** Sketch the graph $r = \theta$ in $\mathbb{R}^2$.

Example 18. Sketch the graph $r = 1 - a\cos\theta$ for
(i) $0 < a < 1$.
(ii) $a = 1$.
(iii) $a > 1$.

Theorem 19. *Let $D \subseteq \mathbb{R}^2$ be a domain such that $(0,0) \in \overline{D}$. Let $f : D \to \mathbb{R}$ be a real-valued function. Then*

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} f(r,\theta)$$

provided that either side exists.

Example 20. Find $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3+xy^3}{x^2+y^2}$, where the domain is $\mathbb{R}^2\backslash\{(0,0)\}$.

Solution. Consider $x = r\cos\theta$ and $y = r\sin\theta$.

$$\begin{aligned}\lim_{r\to 0}\frac{(r\cos\theta)^3 + (r\cos\theta)(r\sin\theta)^3}{(r\cos\theta)^2 + (r\sin\theta)^2} &= \lim_{r\to 0}\frac{r^3\cos^3\theta + r^4\cos\theta\sin^3\theta}{r^2}\\ &= \lim_{r\to 0}\left(r\cos^3\theta + r^2\cos\theta\sin^3\theta\right) = 0.\end{aligned}$$

By Theorem 19, $\displaystyle\lim_{(x,y)\to(0,0)}\frac{x^3 + xy^3}{x^2 + y^2} = 0.$

□

Example 21. Find $\displaystyle\lim_{(x,y)\to(0,0)}\frac{3x^2 - 5y^2}{x^2 + y^2}$, where the domain is $\mathbb{R}^2\backslash\{(0,0)\}$.

Solution. Consider $x = r\cos\theta$ and $y = r\sin\theta$.

$$\begin{aligned}\lim_{r\to 0}\frac{3(r\cos\theta)^2 - 5(r\sin\theta)^2}{(r\cos\theta)^2 + (r\sin\theta)^2} &= \lim_{r\to 0}\frac{3r^2\cos^2\theta - 5r^2\sin^2\theta}{r^2}\\ &= \lim_{r\to 0}\left(3\cos^2\theta - 5\sin^2\theta\right)\end{aligned}$$

which does not exist since it has different value for different θ. By Theorem 19, the desired limit does not exist.

□