MATH 2010E ADVANCED CALCULUS I
LECTURE 5

WING HONG TONY WONG

13.5 — Tangential and normal components of acceleration

Let I be an interval in $\mathbb{R}$, and let $\mathbf{x} : I \to \mathbb{R}^3$ be a smooth curve. Let

$$s(t) = \int_{t_0}^{t} \|\mathbf{x}'(\tau)\| d\tau$$

be the arc length parameter. Let $\widetilde{\mathbf{x}}(s) = \mathbf{x}(t(s))$. Recall that

$$\mathbf{T} = \widetilde{\mathbf{x}}'(s) = \frac{\mathbf{x}'(t)}{\|\mathbf{x}'(t)\|},$$

$$\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\| = \frac{1}{\|\mathbf{x}'(t)\|} \left\| \frac{d\mathbf{T}}{dt} \right\|,$$

and

$$\mathbf{N} = \frac{1}{\kappa} \frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}/dt}{\|d\mathbf{T}/dt\|}.$$

We define the **unit binormal vector** as

$$\mathbf{B} = \mathbf{T} \times \mathbf{N},$$

which measures the tendency of the curve to twist out of the $\mathbf{TN}$-plane. The three vectors $\mathbf{T}, \mathbf{N}, \mathbf{B}$ form the **Frenet frame**. It is a traveling frame, i.e. the frame keeps turning along the curve when t changes.

We can try to use this traveling frame to express other vectors. For example, we know that $\mathbf{x}'(t) = \|\mathbf{x}'(t)\|\mathbf{T}$. Also,

$$\begin{aligned} \mathbf{x}''(t) &= \left(\frac{d}{dt} \|\mathbf{x}'(t)\| \right) \mathbf{T} + \|\mathbf{x}'(t)\| \frac{d\mathbf{T}}{ds} \frac{ds}{dt} \\ &= \left(\frac{d}{dt} \|\mathbf{x}'(t)\| \right) \mathbf{T} + \kappa \|\mathbf{x}'(t)\|^2 \mathbf{N}. \end{aligned}$$

Notice that the acceleration vector $\mathbf{x}''(t)$ has no component in the direction of binormal $\mathbf{B}$. The tangent component of $\mathbf{x}''(t)$ is $\frac{d}{dt}\|\mathbf{x}'(t)\|$, which measures the rate of change of the <u>length</u> of $\mathbf{x}'(t)$; the normal component of $\mathbf{x}'(t)$ is $\kappa\|\mathbf{x}'(t)\|^2$, which measures the rate of change of the <u>direction</u> of $\mathbf{x}'(t)$.

An interesting application of this result is that if a car makes a sharp turn, doubling the speed will require quadrupling the centripetal force. In case the tyres cannot provide such force, the car will skid and is very dangerous. Also, if an object moves in a circle at

Date: Tuesday, 26^{th} May, 2015.

a constant speed, then only the normal direction $\mathbf{N}$ has nonzero acceleration. Finally, to calculate the normal component of $\mathbf{x}''(t)$, we may use the formula

$$\sqrt{\|\mathbf{x}''(t)\|^2 - \left(\frac{d}{dt}\|\mathbf{x}'(t)\|\right)^2},$$

so that we do not need to find κ.

Example 1. Let $\mathbf{x}(t) = (\cos t + t\sin t, \sin t - t\cos t)$, $t > 0$. Find $\mathbf{x}''(t)$ in terms of $\mathbf{T}$ and $\mathbf{N}$, without evaluating $\mathbf{T}$ and $\mathbf{N}$ explicitly.

Solution. $\mathbf{x}'(t) = (-\sin t + \sin t + t\cos t, \cos t - \cos t + t\sin t) = (t\cos t, t\sin t)$, so we have $\|\mathbf{x}'(t)\| = t$. In other words, the coefficient of $\mathbf{T}$ is $\frac{d}{dt}\|\mathbf{x}'(t)\| = 1$.

Next, $\mathbf{x}''(t) = (\cos t - t\sin t, \sin t + t\cos t)$. Hence, the coefficient of $\mathbf{N}$ is

$$\sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 - 1^2} = t.$$

Therefore, $\mathbf{x}''(t) = \mathbf{T} + t\mathbf{N}$.

$\square$

We can also try to use the traveling frame to express $\frac{d\mathbf{B}}{ds}$.

$$\frac{d\mathbf{B}}{ds} = \frac{d\mathbf{T}\times\mathbf{N}}{ds} = \frac{d\mathbf{T}}{ds}\times\mathbf{N} + \mathbf{T}\times\frac{d\mathbf{N}}{ds} = \mathbf{T}\times\frac{d\mathbf{N}}{ds}$$

since $\frac{d\mathbf{T}}{ds}$ and $\mathbf{N}$ are parallel. From the cross product, we know that $\frac{d\mathbf{B}}{ds}$ is orthogonal to $\mathbf{T}$. Moreover, $\frac{d\mathbf{B}}{ds}$ is orthogonal to $\mathbf{B}$ since $\mathbf{B}$ is of constant length. Therefore, $\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$ for some real number τ. This τ is called the **torsion** of the curve, and the negative sign is a tradition. In other words,

$$\tau = -\frac{d\mathbf{B}}{ds}\cdot\mathbf{N}.$$

Unlike the curvature κ which is always nonnegative, the torsion τ can be both positive or negative. The torsion measures the rate at which the osculating plane turns about $\mathbf{T}$. In other words, it measures how the curve twists. A space curve is a helix if and only if both curvature κ and torsion τ are nonzero constants.

Other formulae for curvature κ and torsion τ

$$\begin{aligned}\mathbf{x}'(t)\times\mathbf{x}''(t) &= \|\mathbf{x}'(t)\|\mathbf{T}\times\left[\left(\frac{d}{dt}\|\mathbf{x}'(t)\|\right)\mathbf{T} + \kappa\|\mathbf{x}'(t)\|^2\mathbf{N}\right]\\ &= \kappa\|\mathbf{x}'(t)\|^3\mathbf{T}\times\mathbf{N}\\ &= \kappa\|\mathbf{x}'(t)\|^3\mathbf{B}.\end{aligned}$$

Therefore, $\kappa = \dfrac{\|\mathbf{x}'(t)\times\mathbf{x}''(t)\|}{\|\mathbf{x}'(t)\|^3}$.

$$\tau = -\left(\mathbf{T}\times\frac{d\mathbf{N}}{ds}\right)\cdot\mathbf{N} = \left(\frac{d\mathbf{N}}{ds}\times\mathbf{T}\right)\cdot\mathbf{N}.$$

To proceed, we need to find $\dfrac{\mathbf{dN}}{ds}$.

$$\begin{aligned}
\mathbf{x}''(t) &= \left(\frac{d}{dt}\|\mathbf{x}'(t)\|\right)\mathbf{T} + \kappa\|\mathbf{x}'(t)\|^2\mathbf{N} \\
\mathbf{x}'''(t) &= \left(\frac{d^2}{dt^2}\|\mathbf{x}'(t)\|\right)\mathbf{T} + \left(\frac{d}{dt}\|\mathbf{x}'(t)\|\right)\frac{\mathbf{dT}}{dt} + \frac{d\kappa\|\mathbf{x}'(t)\|^2}{dt}\mathbf{N} + \kappa\|\mathbf{x}'(t)\|^2\frac{d\mathbf{N}}{dt} \\
\frac{d\mathbf{N}}{dt} &= \frac{1}{\kappa\|\mathbf{x}'(t)\|^2}\left[\mathbf{x}'''(t) - \left(\frac{d^2}{dt^2}\|\mathbf{x}'(t)\|\right)\mathbf{T} - \left(\frac{d}{dt}\|\mathbf{x}'(t)\|\right)\frac{\mathbf{dT}}{dt} - \frac{d\kappa\|\mathbf{x}'(t)\|^2}{dt}\mathbf{N}\right] \\
\frac{d\mathbf{N}}{ds}\frac{ds}{dt} &= \frac{1}{\kappa\|\mathbf{x}'(t)\|^2}\left[\mathbf{x}'''(t) - \left(\frac{d^2}{dt^2}\|\mathbf{x}'(t)\|\right)\mathbf{T} - \left(\frac{d}{dt}\|\mathbf{x}'(t)\|\right)\frac{\mathbf{dT}}{dt} - \frac{d\kappa\|\mathbf{x}'(t)\|^2}{dt}\mathbf{N}\right] \\
\frac{d\mathbf{N}}{ds} &= \frac{1}{\kappa\|\mathbf{x}'(t)\|^3}\left[\mathbf{x}'''(t) - \left(\frac{d^2}{dt^2}\|\mathbf{x}'(t)\|\right)\mathbf{T} - \left(\frac{d}{dt}\|\mathbf{x}'(t)\|\right)\frac{\mathbf{dT}}{dt} - \frac{d\kappa\|\mathbf{x}'(t)\|^2}{dt}\mathbf{N}\right].
\end{aligned}$$

Hence,

$$\frac{d\mathbf{N}}{ds}\times\mathbf{T} = \frac{1}{\kappa\|\mathbf{x}'(t)\|^3}\left[\mathbf{x}'''(t)\times\mathbf{T} - \left(\frac{d}{dt}\|\mathbf{x}'(t)\|\right)\frac{\mathbf{dT}}{dt}\times\mathbf{T} - \frac{d\kappa\|\mathbf{x}'(t)\|^2}{dt}\mathbf{N}\times\mathbf{T}\right]$$

since $\mathbf{T}\times\mathbf{T}=\mathbf{0}$. Also, since $\dfrac{d\mathbf{T}}{dt} = \left\|\dfrac{d\mathbf{T}}{dt}\right\|\mathbf{N}$, we have

$$\left(\frac{d\mathbf{T}}{dt}\times\mathbf{T}\right)\cdot\mathbf{N} = 0 \text{ and } (\mathbf{N}\times\mathbf{T})\cdot\mathbf{N} = 0.$$

As a result,

$$\begin{aligned}
\tau = \left(\frac{d\mathbf{N}}{ds}\times\mathbf{T}\right)\cdot\mathbf{N} &= \frac{1}{\kappa\|\mathbf{x}'(t)\|^3}\left(\mathbf{x}'''(t)\times\mathbf{T}\right)\cdot\mathbf{N} \\
&= \frac{1}{\kappa\|\mathbf{x}'(t)\|^4}\left(\mathbf{x}'''(t)\times\mathbf{x}'(t)\right)\cdot\frac{1}{\kappa}\frac{d\mathbf{T}}{ds} \\
&= \frac{1}{\kappa^2\|\mathbf{x}'(t)\|^4}\left(\mathbf{x}'''(t)\times\mathbf{x}'(t)\right)\cdot\frac{d\mathbf{T}}{dt}\frac{dt}{ds} \\
&= \frac{1}{\kappa^2\|\mathbf{x}'(t)\|^5}\left(\mathbf{x}'''(t)\times\mathbf{x}'(t)\right)\cdot\frac{d}{dt}\frac{\mathbf{x}'(t)}{\|\mathbf{x}'(t)\|} \\
&= \frac{1}{\kappa^2\|\mathbf{x}'(t)\|^5}\left(\mathbf{x}'''(t)\times\mathbf{x}'(t)\right)\cdot\left[\left(\frac{d}{dt}\frac{1}{\|\mathbf{x}'(t)\|}\right)\mathbf{x}'(t) + \frac{\mathbf{x}''(t)}{\|\mathbf{x}'(t)\|}\right] \\
&= \frac{1}{\kappa^2\|\mathbf{x}'(t)\|^6}\left(\mathbf{x}'''(t)\times\mathbf{x}'(t)\right)\cdot\mathbf{x}''(t) \\
&= \frac{\left(\mathbf{x}'''(t)\times\mathbf{x}'(t)\right)\cdot\mathbf{x}''(t)}{\|\mathbf{x}'(t)\times\mathbf{x}''(t)\|^2} \\
&= \frac{\begin{vmatrix} x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \\ x'''(t) & y'''(t) & z'''(t) \end{vmatrix}}{\|\mathbf{x}'(t)\times\mathbf{x}''(t)\|^2}
\end{aligned}$$

if $\|\mathbf{x}'(t)\times\mathbf{x}''(t)\| \neq 0$.

Example 2. Let $\mathbf{x}(t) = (a\cos t, a\sin t, bt)$, $a, b \geq 0$, $a^2 + b^2 \neq 0$. Find the curvature κ and torsion τ.

Solution. First, we have $\mathbf{x}'(t) = (-a\sin t, a\cos t, b)$, $\mathbf{x}''(t) = (-a\cos t, -a\sin t, 0)$, and $\mathbf{x}'''(t) = (a\sin t, -a\cos t, 0)$, so

$$\|\mathbf{x}'(t)\| = \sqrt{(-a\sin t)^2 + (a\cos t)^2 + b^2} = \sqrt{a^2 + b^2},$$

and

$$\begin{aligned}
\|\mathbf{x}'(t) \times \mathbf{x}''(t)\| &= \left\| \begin{vmatrix} \widehat{\mathbf{i}} & \widehat{\mathbf{j}} & \widehat{\mathbf{k}} \\ -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \end{vmatrix} \right\| \\
&= \sqrt{(ab\sin t)^2 + (-ab\cos t)^2 + (a^2\sin^2 t + a^2\cos^2 t)^2} \\
&= a\sqrt{a^2 + b^2}.
\end{aligned}$$

Therefore,

$$\kappa = \frac{\|\mathbf{x}'(t) \times \mathbf{x}''(t)\|}{\|\mathbf{x}'(t)\|^3} = \frac{a}{a^2 + b^2},$$

and

$$\tau = \frac{\begin{vmatrix} -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \\ a\sin t & -a\cos t & 0 \end{vmatrix}}{\left(a\sqrt{a^2 + b^2}\right)^2} = \frac{b(a^2\cos^2 t + a^2\sin^2 t)}{a^2(a^2 + b^2)} = \frac{b}{a^2 + b^2}.$$

□

13.6 — Velocity and acceleration in polar coordinates

If $\mathbf{x}$ is a curve in two-dimensional polar coordinates, then we can use a different traveling frame from the Frenet frame. For each point $\mathbf{x}(t) = r(t)\,(\cos\theta(t), \sin\theta(t))$, let

$$\mathbf{u}_r = (\cos\theta, \sin\theta) \quad \text{and} \quad \mathbf{u}_\theta = (-\sin\theta, \cos\theta).$$

Then

$$\frac{d\mathbf{u}_r}{dt} = \frac{d\mathbf{u}_r}{d\theta}\frac{d\theta}{dt} = (-\sin\theta, \cos\theta)\theta'(t) = \theta'(t)\mathbf{u}_\theta,$$

and

$$\frac{d\mathbf{u}_\theta}{dt} = \frac{d\mathbf{u}_\theta}{d\theta}\frac{d\theta}{dt} = (-\cos\theta, -\sin\theta)\theta'(t) = -\theta'(t)\mathbf{u}_r.$$

Therefore,

$$\mathbf{x}(t) = r(t)\mathbf{u}_r,$$
$$\mathbf{x}'(t) = r'(t)\mathbf{u}_r + r(t)\theta'(t)\mathbf{u}_\theta,$$

and

$$\begin{aligned}
\mathbf{x}''(t) &= r''(t)\mathbf{u}_r + r'(t)\theta'(t)\mathbf{u}_\theta + r'(t)\theta'(t)\mathbf{u}_\theta + r(t)\theta''(t)\mathbf{u}_\theta + r(t)\theta'(t)(-\theta'(t)\mathbf{u}_r) \\
&= (r''(t) - r(t)\theta'(t)^2)\mathbf{u}_r + (2r'(t)\theta'(t) + r(t)\theta''(t))\mathbf{u}_\theta.
\end{aligned}$$

If $\mathbf{x}$ is a curve in a three-dimensional polar coordinates, i.e.

$$\mathbf{x}(t) = \left(r(t)\cos\theta(t), r(t)\sin\theta(t), z(t)\right),$$

then we let

$$\mathbf{u}_r = (\cos\theta, \sin\theta, 0), \quad \mathbf{u}_\theta = (-\sin\theta, \cos\theta, 0), \quad \text{and} \quad \mathbf{k} = (0, 0, 1),$$

so that $\mathbf{k} = \mathbf{u}_r \times \mathbf{u}_\theta$. In this case,

$$\mathbf{x}(t) = r(t)\mathbf{u}_r + z(t)\mathbf{k},$$
$$\mathbf{x}'(t) = r'(t)\mathbf{u}_r + r(t)\theta'(t)\mathbf{u}_\theta + z'(t)\mathbf{k},$$

and

$$\mathbf{x}''(t) = (r''(t) - r(t)\theta'(t)^2)\mathbf{u}_r + (2r'(t)\theta'(t) + r(t)\theta''(t))\mathbf{u}_\theta + z''(t)\mathbf{k}.$$

Planet movement

Let M be a mass at the origin, and let m be a mass moving around M. Let $\mathbf{x}(t)$ be the position of m. Let G be the **universal gravitation constant**. Then by Newton's law of gravitation, the gravitational force between M and m is

$$-\frac{GmM\mathbf{x}}{\|\mathbf{x}\|^3}.$$

Hence, the acceleration of m is

$$\mathbf{x}''(t) = -\frac{GM\mathbf{x}}{\|\mathbf{x}\|^3}.$$

Proposition 3. $\mathbf{x}(t) \times \mathbf{x}'(t)$ *is a constant.*

Proof. $\frac{d}{dt}\mathbf{x}(t) \times \mathbf{x}'(t) = \mathbf{x}'(t) \times \mathbf{x}'(t) + \mathbf{x}(t) \times \mathbf{x}''(t) = \mathbf{0}$ since $\mathbf{x}''(t) = -\frac{GM\mathbf{x}}{\|\mathbf{x}\|^3}$.

□

Proposition 3 shows that planet movements are on a fixed plane. We are going to skip deriving the Kepler's law of motion in this course.

14.1 — Functions of several variables

Let $B_\epsilon(\mathbf{x}_0) = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{x}_0\| < \epsilon\}$ be an **open ϵ-ball around $\mathbf{x}_0$** in $\mathbb{R}^n$, and let $\overline{B_\epsilon(\mathbf{x}_0)} = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{x}_0\| \le \epsilon\}$ be a **closed ϵ-ball around $\mathbf{x}_0$**.

Let $S \subseteq \mathbb{R}^n$. Then for any point $\mathbf{x} \in \mathbb{R}^n$, it can be classified into one and only one of the followings.

- $\mathbf{x}$ is an **interior point** of S if there exists $\epsilon > 0$ such that $B_\epsilon(\mathbf{x}) \subseteq S$.
- $\mathbf{x}$ is an **exterior point** of S if there exists $\epsilon > 0$ such that $B_\epsilon(x) \cap S = \emptyset$.
- $\mathbf{x}$ is a **boundary point** of S if for all $\epsilon > 0$, $B_\epsilon(\mathbf{x}) \nsubseteq S$ and $B_\epsilon(\mathbf{x}) \cap S \neq \emptyset$.

Please note that if $\mathbf{x}$ is a boundary point of S, then x may be in S and may be not in S. We let $\text{int}(S)$ be the set of all interior points of S, $\text{ext}(S)$ the set of all exterior points of S, and ∂S the set of all boundary points of S.

S is **open** if one of the following holds.

- $S = \text{int}(S)$.
- $\partial S \cap S = \emptyset$.
- For all $\mathbf{x} \in S$, there exists $\epsilon > 0$ such that $B_\epsilon(\mathbf{x}) \subseteq S$.

S is **closed** if one of the following holds.

- $S = \text{int}(S) \cup \partial S$.
- $\partial S \subseteq S$.
- $\mathbb{R}^n \backslash S$ is open.

Warning: A set S is not open does NOT imply that S is closed, vice versa. In other words, S can be neither open nor closed.

A set S is both open AND closed in $\mathbb{R}^n$ if and only if $S = \emptyset$ or $S = \mathbb{R}^n$.

Here are a couple more definitions.

- S is **bounded** if there exists $R > 0$ such that $S \subseteq B_R(\mathbf{0})$.
- S is **compact** if S is closed and bounded.
- S is **path-connected** if for every two points $\mathbf{a}$ and $\mathbf{b}$ in S, there exists a continuous path $\mathbf{x} : [a, b] \to S$ such that $\mathbf{x}(a) = \mathbf{a}$ and $\mathbf{x}(b) = \mathbf{b}$.
- S is a **domain** if S is open and path-connected.

Let $D \subseteq \mathbb{R}^n$ be a domain. A **real-valued function** $f : D \to \mathbb{R}$ is defined such that $y = f(x_1, x_2, \ldots, x_n) \in \mathbb{R}$, where y is a **dependent variable**, while $x_1, x_2, \ldots, x_n$ are **independent variables**.

Example 4. Specify the largest domains and ranges of f.
(a) $f(x, y) = \sqrt{y - x^2}$.
(b) $f(x, y) = \dfrac{1}{xy}$.
(c) $f(x, y) = \sin xy$.
(d) $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$
(e) $f(x, y, z) = \dfrac{1}{x^2 + y^2 + z^2}$
(f) $f(x, y, z) = xy \ln z$.

Example 5. Specify the largest domain of $F(x, y) = \big(\ln(x - y^2), \ln(y - x^2)\big)$.

If the real-valued function f has only two independent variables, we often write it as $z = f(x, y)$. It can be graphed as a surface in $\mathbb{R}^3$ over the domain D, sometimes with the aid of **contours (level curves)** $f(x, y) = c$ in $\mathbb{R}^2$.

Example 6. Graph $f(x, y) = 100 - x^2 - y^2$, and graph the level curves $f(x, y) = 0$, $f(x, y) = 51$, $f(x, y) = 75$, and $f(x, y) = -44$.

Example 7. Graph $f(x, y) = x^2 - y^2$, and graph the level curves $f(x, y) = 0$, $f(x, y) = 1$, and $f(x, y) = -1$.

Solution. Recall from Section 12.6 that $z = x^2 - y^2$ is a hyperbolic paraboloid. The "vertical slicing" $x = 0$ and $y = 0$ gives parabolas, while "horizontal slicing" $x^2 - y^2 = c$ gives the hyperbolic contours (level curves).

$\square$

If the real-valued function f has three independent variables, we usually try to understand f by graphing the **level surfaces** $f(x, y, z) = c$ in $\mathbb{R}^3$.

Example 8. Graph the level surfaces of $f(x, y, z) = x^2 + y^2 - z^2$.