MATH 2010E ADVANCED CALCULUS I
LECTURE 4

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13.2 — Integrals of vector functions

Let I be an interval in $\mathbb{R}$, and let $\mathbf{x} : I \to \mathbb{R}^m$ be a curve in $\mathbb{R}^m$. Let
$$\mathbf{x}(t) = \big(x_1(t), x_2(t), \dots, x_m(t)\big).$$
Then
$$\int \mathbf{x}(t)dt = \left(\int x_1(t)dt, \int x_2(t)dt, \dots, \int x_m(t)dt\right)$$
if either side exists. If the antiderivative of each component is $\displaystyle\int x_i(t) = X_i(t) + C_i$, then
$$\int \mathbf{x}(t)dt = \big(X_1(t), X_2(t), \dots, X_m(t)\big) + \mathbf{C}.$$
For **definite integrals**,
$$\begin{aligned}\int_a^b \mathbf{x}(t)dt &= \left(\int_a^b x_1(t)dt, \int_a^b x_2(t)dt, \dots, \int_a^b x_m(t)dt\right)\\ &= \big(X_1(b), X_2(b), \dots, X_m(b)\big) - \big(X_1(a), X_2(a), \dots, X_m(a)\big).\end{aligned}$$

13.2 — Projectile motion

Consider a projectile motion in $\mathbb{R}^2$. Let the initial position of the projectile motion be the origin $\mathbf{0}$, and let the initial velocity vector be $\mathbf{v}_0$. The gravity asserts a constant acceleration $\mathbf{a} = -g\widehat{\mathbf{j}}$. In other words, if the projectile is given by $\mathbf{x}(t)$, where $t \in [0, \infty)$, then
$$\mathbf{x}(0) = \mathbf{0}, \quad \mathbf{x}'(0) = \mathbf{v}_0, \quad \mathbf{x}''(t) = -g\widehat{\mathbf{j}}.$$

Hence,
$$\mathbf{x}'(t) = \int -g\widehat{\mathbf{j}}dt = -gt\widehat{\mathbf{j}} + \mathbf{C}.$$
By $\mathbf{x}'(0) = \mathbf{v}_0$, we have $\mathbf{C} = \mathbf{v}_0$. Now,
$$\mathbf{x}(t) = \int -gt\widehat{\mathbf{j}} + \mathbf{v}_0 = -\frac{1}{2}gt^2\widehat{\mathbf{j}} + \mathbf{v}_0 t + \widetilde{\mathbf{C}}.$$
By $\mathbf{x}(0) = \mathbf{0}$, we have $\widetilde{\mathbf{C}} = \mathbf{0}$. Therefore,
$$\mathbf{x}(t) = -\frac{1}{2}gt^2\widehat{\mathbf{j}} + \mathbf{v}_0 t = (\|\mathbf{v}_0\| \cos\alpha)t\widehat{\mathbf{i}} + \left((\|\mathbf{v}_0\| \sin\alpha)t - \frac{1}{2}gt^2\right)\widehat{\mathbf{j}},$$
where α is the angle between $\mathbf{v}_0$ and the x-axis.

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With this formula, we can obtain the following results.

- Maximum height

  $x_2'(t) = \|\mathbf{v}_0\| \sin\alpha - gt = 0$ implies $t = \dfrac{\|\mathbf{v}_0\| \sin\alpha}{g}$. Since $x_2''(t) = -g < 0$, we know that $x_2\left(\dfrac{\|\mathbf{v}_0\| \sin\alpha}{g}\right) = \dfrac{(\|\mathbf{v}_0\| \sin\alpha)^2}{2g}$ is the global maximum.
- Flight time

  $x_2(t) = 0$ implies $t = 0$ or $t = \dfrac{2\|\mathbf{v}_0\| \sin\alpha}{g}$, where the second value gives the flight time.
- Range

  At the end of the flight time, $x_1\left(\dfrac{2\|\mathbf{v}_0\| \sin\alpha}{g}\right) = \dfrac{\|\mathbf{v}_0\|^2 \sin 2\alpha}{g}$.

13.3 — Arc length of a curve

Let I be an interval in $\mathbb{R}$, and let $\mathbf{x} : I \to \mathbb{R}^m$ be a smooth curve in $\mathbb{R}^m$. The arc length of the curve for $t \in [a, b]$ is

$$\int_a^b \|\mathbf{x}'(t)\| dt = \int_a^b \sqrt{\big(x_1'(t)\big)^2 + \big(x_2'(t)\big)^2 + \cdots + \big(x_m'(t)\big)^2} dt.$$

Example 1. Let $\mathbf{x}(t) = (\cos t, \sin t, t)$, $t \in [0, 2\pi]$ be the helix curve.
(a) Find the tangent line at $t = \pi$.
(b) Find the length of the curve.

Solution. (a) The tangent line is

$$\begin{aligned}\{\mathbf{x}(\pi) + t\mathbf{x}'(\pi) : t \in \mathbb{R}\} &= \{(\cos\pi, \sin\pi, \pi) + t(-\sin\pi, \cos\pi, 1) : t \in \mathbb{R}\} \\ &= \{(-1, 0, \pi) + t(0, -1, 1) : t \in \mathbb{R}\}.\end{aligned}$$

(b) The length is

$$\int_0^{2\pi} \|(-\sin t, \cos t, 1)\| dt = \int_0^{2\pi} \sqrt{(-\sin t)^2 + \cos^2 t + 1}\ dt = \sqrt{2}t\Big|_0^{2\pi} = 2\sqrt{2}\pi.$$

□

Given a smooth curve $\mathbf{x}(t)$, it will be very convenient if we can reparametrize the curve so that all tangent vectors are of unit length. Such a parametrization is called the **arc length parametrization**, given by

$$s(t) = \int_{t_0}^t \|\mathbf{x}'(\tau)\| d\tau = \int_{t_0}^t \sqrt{\big(x_1'(\tau)\big)^2 + \big(x_2'(\tau)\big)^2 + \cdots + \big(x_m'(\tau)\big)^2} d\tau.$$

Example 2. Reparametrize the curve $\mathbf{x}(t) = (\cos t, \sin t, t)$ using the arc length parameter.

Solution. Let $t_0 = 0$.

$$s(t) = \int_0^t \sqrt{(-\sin\tau)^2 + \cos^2\tau + 1}\, d\tau = \sqrt{2}t.$$

In other words, $t = \frac{\sqrt{2}s}{2}$. Hence,

$$\widetilde{\mathbf{x}}(s) = \mathbf{x}(t(s)) = \left(\cos\frac{\sqrt{2}s}{2}, \sin\frac{\sqrt{2}s}{2}, \frac{\sqrt{2}s}{2}\right).$$

□

Since $\mathbf{x}$ is a smooth curve, $\mathbf{x}'$ is continuous by definition of smoothness, implying that $\|\mathbf{x}'(\tau)\|$ is also continuous. By Fundamental Theorem of Calculus (of scalar valued functions),

$$\frac{ds}{dt} = \|\mathbf{x}'(t)\|.$$

Therefore,

$$\begin{aligned}
\widetilde{\mathbf{x}}'(s) &= \frac{d\widetilde{\mathbf{x}}(s)}{ds} \\
&= \frac{d\mathbf{x}(t(s))}{ds} \\
&= \left(\frac{dx_1(t(s))}{ds}, \frac{dx_2(t(s))}{ds}, \dots, \frac{dx_m(t(s))}{ds}\right) \\
&= \left(\frac{dx_1(t)}{dt}\frac{dt}{ds}, \frac{dx_2(t)}{dt}\frac{dt}{ds}, \dots, \frac{dx_m(t)}{dt}\frac{dt}{ds}\right) \text{ (by chain rule of scalar valued functions)} \\
&= \left(\frac{dx_1(t)}{dt}, \frac{dx_2(t)}{dt}, \dots, \frac{dx_m(t)}{dt}\right)\frac{dt}{ds} \\
&= \mathbf{x}'(t)\frac{1}{ds/dt} \\
&= \frac{\mathbf{x}'(t)}{\|\mathbf{x}'(t)\|}.
\end{aligned}$$

This shows that all tangent vectors of $\widetilde{\mathbf{x}}(s)$ are of unit length.

For the convenience of future discussions, we let $\mathbf{T} = \frac{\mathbf{x}'(t)}{\|\mathbf{x}'(t)\|} = \widetilde{\mathbf{x}}'(s)$ be the **unit tangent vector**.

13.4 — Curvature and normal vectors of a curve

Let I be an interval in $\mathbb{R}$, and let $\mathbf{x} : I \to \mathbb{R}^m$ be a <u>smooth</u> curve in $\mathbb{R}^m$. The **curvature** of the curve is defined as

$$\kappa = \left\|\frac{d\mathbf{T}}{ds}\right\|.$$

If the curve $\mathbf{x}$ is not parametrized using the arc length parameter, then

$$\begin{aligned}\kappa &= \left\|\frac{d\mathbf{T}}{ds}\right\| \\ &= \left\|\frac{d\mathbf{T}}{dt}\frac{dt}{ds}\right\| \text{ (by chain rule)} \\ &= \left\|\frac{d\mathbf{T}}{dt}\right\|\frac{1}{|ds/dt|} \\ &= \frac{1}{\|\mathbf{x}'(t)\|}\left\|\frac{d\mathbf{T}}{dt}\right\|.\end{aligned}$$

Example 3. $\mathbf{x}(t) = \mathbf{p} + t\mathbf{v}$, $t \in \mathbb{R}$ is a straight line. Find the curvature κ.

Solution. $\mathbf{T} = \dfrac{\mathbf{x}'(t)}{\|\mathbf{x}'(t)\|} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ is a constant vector independent of t. Hence, the curvature is

$$\kappa = \frac{1}{\|\mathbf{v}\|}\left\|\frac{d(\mathbf{v}/\|\mathbf{v}\|)}{dt}\right\| = \frac{1}{\|\mathbf{v}\|}\|\mathbf{0}\| = 0.$$

□

Example 4. Let $\mathbf{x}(t) = (r\cos t, r\sin t)$, $t \in \mathbb{R}$. Find the curvature κ.

Solution. $\mathbf{T} = \dfrac{\mathbf{x}'(t)}{\|\mathbf{x}'(t)\|} = \dfrac{(-r\sin t, r\cos t)}{\|(-r\sin t, r\cos t)\|} = \dfrac{(-r\sin t, r\cos t)}{r} = (-\sin t, \cos t)$. Hence, the curvature is

$$\kappa = \frac{1}{\|(-r\sin t, r\cos t)\|}\left\|\frac{d(-\sin t, \cos t)}{dt}\right\| = \frac{1}{r}\|(-\cos t, -\sin t)\| = \frac{1}{r}.$$

□

The result in Example 4 makes sense in the way that when r is huge, the circle is almost like a straight line, hence κ is close to 0.

The **principal unit normal vector** of a smooth curve $\mathbf{x}$ is

$$\mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds}.$$

This vector $\mathbf{N}$ always point to the direction where $\mathbf{T}$ turns. $\mathbf{N}$ is normal to $\mathbf{T}$ since

$$\begin{aligned}\mathbf{T}\cdot\mathbf{T} &= 1 \\ \frac{d}{ds}(\mathbf{T}\cdot\mathbf{T}) &= 0 \\ \frac{d\mathbf{T}}{ds}\cdot\mathbf{T} + \mathbf{T}\cdot\frac{d\mathbf{T}}{ds} &= 0 \\ 2\mathbf{T}\cdot\frac{d\mathbf{T}}{ds} &= 0 \\ \mathbf{T}\cdot\frac{d\mathbf{T}}{ds} &= 0.\end{aligned}$$

If the curve $\mathbf{x}$ is not parametrized using the arc length parameter, then

$$\begin{aligned}\mathbf{N} &= \frac{1}{\kappa}\frac{d\mathbf{T}}{ds} \\ &= \|\mathbf{x}'(t)\|\frac{1}{\|d\mathbf{T}/dt\|}\frac{d\mathbf{T}}{dt}\frac{dt}{ds} \\ &= \|\mathbf{x}'(t)\|\frac{1}{\|d\mathbf{T}/dt\|}\frac{d\mathbf{T}}{dt}\frac{1}{\|\mathbf{x}'(t)\|} \\ &= \frac{d\mathbf{T}/dt}{\|d\mathbf{T}/dt\|}.\end{aligned}$$

Example 5. Let $\mathbf{x}(t) = (\cos 2t, \sin 2t)$, $t \in \mathbb{R}$. Find $\mathbf{T}$ and $\mathbf{N}$.

Solution. $\mathbf{x}'(t) = (-2\sin 2t, 2\cos 2t)$, $\|\mathbf{x}'(t)\| = 2$, so

$$\mathbf{T} = \frac{(-2\sin 2t, 2\cos 2t)}{2} = (-\sin 2t, \cos 2t).$$

$\dfrac{d\mathbf{T}}{dt} = (-2\cos 2t, -2\sin 2t)$, $\left\|\dfrac{d\mathbf{T}}{dt}\right\| = 2$, so

$$\mathbf{N} = \frac{(-2\cos 2t, -2\sin 2t)}{2} = (-\cos 2t, -\sin 2t).$$

□

The **circle of curvature** or **osculating circle** at a point $\mathbf{p}$ on a planar curve $\mathbf{x}$ with $\kappa \neq 0$ is the circle that satisfies the following conditions.

- The circle is tangent to $\mathbf{x}$ at $\mathbf{p}$, i.e. has the same tangent line as the curve.
- The circle has the same curvature as the curve at $\mathbf{p}$.
- The center of the circle lies on the same side as $\mathbf{N}$.

The **radius of curvature** ρ is the radius of the circle of curvature, satisfying

$$\rho = \frac{1}{\kappa}$$

by Example 4.

Example 6. Find the circle of curvature of the parabola $y = x^2$ at the origin.

Solution. A parametrization of the curve is $\mathbf{x}(t) = (t, t^2)$. Then $\mathbf{x}'(t) = (1, 2t)$, and $\|\mathbf{x}'(t)\| = \sqrt{1+4t^2}$, so

$$\begin{aligned}\mathbf{T} &= \left(\frac{1}{\sqrt{1+4t^2}}, \frac{2t}{\sqrt{1+4t^2}}\right), \\ \frac{d\mathbf{T}}{dt} &= \left(-\frac{4t}{\left(\sqrt{1+4t^2}\right)^3}, \frac{2+8t^2-8t^2}{\left(\sqrt{1+4t^2}\right)^3}\right) = \left(-\frac{4t}{\left(\sqrt{1+4t^2}\right)^3}, \frac{2}{\left(\sqrt{1+4t^2}\right)^3}\right), \\ \left\|\frac{d\mathbf{T}}{dt}\right\| &= \sqrt{\frac{16t^2+4}{(1+4t^2)^3}} = \frac{2}{\sqrt{1+4t^2}}, \\ \mathbf{N} &= \frac{d\mathbf{T}/dt}{\|d\mathbf{T}/dt\|} = \left(-\frac{2t}{1+4t^2}, \frac{1}{1+4t^2}\right), \\ \kappa &= \frac{1}{\|\mathbf{x}'(t)\|}\left\|\frac{d\mathbf{T}}{dt}\right\| = \frac{1}{\sqrt{1+4t^2}}\frac{2}{\sqrt{1+4t^2}} = \frac{2}{1+4t^2}.\end{aligned}$$

At the origin, $t = 0$. The curvature at the origin is $\kappa(0) = 2$, and the principal unit normal vector is $\mathbf{N}(0) = (0, 1)$. Therefore, the circle of curvature is tangent to the x-axis, has the center on the positive y-axis, and the radius of curvature is $\rho(0) = \dfrac{1}{\kappa(0)} = \dfrac{1}{2}$. In other words, the circle of curvature is

$$x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}.$$

□