MATH 2010E ADVANCED CALCULUS I
LECTURE 3

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12.5 — Lines and planes in $\mathbb{R}^3$

Example 1. Let $Ax + By + Cz = D_1$ and $Ax + By + Cz = D_2$ be two planes in $\mathbb{R}^3$. Find the perpendicular distance between them.

Proof. First, we notice that these two planes are parallel since they share the same normal vector; otherwise, these two planes will intersect and the perpendicular distance is 0.

$\{t(A, B, C) : t \in \mathbb{R}\}$ is a line perpendicular to both planes. The intersections of this line with the two planes are

$$\frac{D_1}{A^2 + B^2 + C^2}(A, B, C) \text{ and } \frac{D_2}{A^2 + B^2 + C^2}(A, B, C),$$

and the desired perpendicular distance is the distance between these two points, which is

$$\frac{\sqrt{(D_1 - D_2)^2 A^2 + (D_1 - D_2)^2 B^2 + (D_1 - D_2)^2 C^2}}{A^2 + B^2 + C^2} = \frac{|D_1 - D_2|}{\sqrt{A^2 + B^2 + C^2}}.$$

□

12.6 — Cylinders and Quadric Surfaces

In $\mathbb{R}^2$, the most general form of a quadratic equation is given by

$$Ax^2 + 2Bxy + Cy^2 + Dx + Ey = F.$$

There are three major types of quadric curves, also known as conic sections.

- Ellipse: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = F$.
  - If $F = 1$, or $F > 0$, then it is nondegenerated.
  - if $F = 0$, then it is a single point.
  - If $F < 0$, then it is empty.
- Hyperbola: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = F$.
  - If $F = 1$, or $F > 0$, then it is nondegenerated with left-right branches.
  - if $F = 0$, then it is degenerated into a "cross" at the origin.
  - If $F = -1$, or $F < 0$, then it is nondegenerated with top-bottom branches.
- Parabola: $y = ax^2$.

All quadratic equations of two variables can be transformed into one of the above forms by change of variables.

Example 2. Rewrite $x^2 + 2xy + 3y^2 + 4x + 5y = 6$ as one of the above forms.

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Solution.

$$\begin{aligned}
x^2+2xy+3y^2+4x+5y&=6\\
(x^2+2xy+y^2)+2y^2+4x+5y&=6\\
(x+y)^2+\left(\sqrt{2}y\right)^2+4(x+y)+\frac{1}{\sqrt{2}}\left(\sqrt{2}y\right)&=6\\
\left[(x+y)^2+4(x+y)+4\right]+\left[\left(\sqrt{2}y\right)^2+2\frac{1}{2\sqrt{2}}\left(\sqrt{2}y\right)+\left(\frac{1}{2\sqrt{2}}\right)^2\right]&=10+\frac{1}{8}\\
(x+y+2)^2+\left(\sqrt{2}y+\frac{1}{2\sqrt{2}}\right)^2&=\frac{81}{8}
\end{aligned}$$

Hence, if we let $u=x+y+2$ and $v=\sqrt{2}y+\frac{1}{2\sqrt{2}}$, then we get a circle. In other words, the original quadratic equation corresponds to an ellipse.

□

In order to determine which type of quadric object a given quadratic equation represents, we only need to study the quadratic portion (not the linear portion) of the equation. Linear algebra can help us to study systematically.

The quadratic expression $Ax^2+2Bxy+Cy^2$ can be written as a matrix multiplication

$$\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} A & B\\ B & C\end{pmatrix}\begin{pmatrix} x\\ y\end{pmatrix}=\mathbf{x}^\top M\mathbf{x}.$$

This expression can be easily generalized to higher dimension by taking $\mathbf{x}=(x_1\ x_2\ \cdots\ x_n)^\top$ and M to be an $n\times n$ **symmetric matrix**.

Let D_i denote the **i-th leading principal minor** of M, i.e. the determinant of the $i\times i$ submatrix formed by the first i rows and the first i columns of M. What we have done to the quadratic portion in Example 2 is

$$\begin{aligned}
\mathbf{x}^\top M\mathbf{x}&=Ax^2+2Bxy+Cy^2\\
&=A\left(x+\frac{B}{A}y\right)^2+\frac{AC-B^2}{A}y^2\\
&=D_1u^2+\frac{D_2}{D_1}v^2\\
&=\begin{pmatrix} u & v\end{pmatrix}\begin{pmatrix} D_1 & 0\\ 0 & \frac{D_2}{D_1}\end{pmatrix}\begin{pmatrix} u\\ v\end{pmatrix}\\
&=\mathbf{u}^\top\Delta\mathbf{u},
\end{aligned}$$

where

$$\mathbf{u}=\begin{pmatrix} u\\ v\end{pmatrix}=\begin{pmatrix} 1 & \frac{B}{A}\\ 0 & 1\end{pmatrix}\begin{pmatrix} x\\ y\end{pmatrix}=U\mathbf{x}.$$

Lagrange noticed that this technique can be generalized to higher dimension. First, we permute the variables $x_1,x_2,\dots,x_n$ and the corresponding rows and columns of M such that $D_1,D_2\dots,D_k\neq 0$ and $D_{k+1},D_{k+2},\dots,D_n=0$ for some $k=1,2,\dots,n$. Next, we obtain the upper triangular matrix U as the **row echelon form** of M with diagonal entries being 1, i.e. no backward substitution needed. Then we have

$$\mathbf{x}^\top M\mathbf{x} = \begin{pmatrix} u_1 & u_2 & \cdots & u_n \end{pmatrix} \begin{pmatrix} D_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \frac{D_2}{D_1} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \frac{D_k}{D_{k-1}} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_k \\ u_{k+1} \\ \vdots \\ u_n \end{pmatrix} = \mathbf{u}^\top \Delta \mathbf{u},$$

where $\mathbf{u} = U\mathbf{x}$. This is because $M = U^\top \Delta U$ by Cholesky decomposition of symmetric matrices.

The type of quadric object that the quadratic equation represents is solely determined by the numbers of zeros, positive and negative entries on the diagonal of Δ. In fact, these numbers are identical to those of the eigenvalues of M.

Given an $n \times n$ matrix A, an **eigenvector** of A is a vector $\mathbf{v} \neq \mathbf{0}$ such that $A\mathbf{v} = \lambda\mathbf{v}$ for some real number λ. Such a real number λ is the **eigenvalue** of A corresponding to $\mathbf{v}$.

In order to find all eigenvalues of A, note that $(A - \lambda I)\mathbf{v} = \mathbf{0}$ has nontrivial solutions, i.e. $A - \lambda I$ is singular. Hence, all eigenvalues λ are solutions to the equation $\det(A - \lambda I) = 0$.

Example 3. Find the eigenvalues of $A = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$.

Solution. $\left| \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = \begin{vmatrix} 1-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 2 = 0$ yields $\lambda = 2 \pm \sqrt{2}$. Hence, the eigenvalues of A are $2 + \sqrt{2}$ and $2 - \sqrt{2}$.

$\square$

As M is symmetric, M is orthogonally diagonalizable with real eigenvalues, i.e. there exist an orthogonal matrix Q and a diagonal matrix Λ (all its diagonal entries are the eigenvalues of M) such that $Q^\top MQ = \Lambda$. In other words, $\mathbf{x}^\top M\mathbf{x} = \mathbf{u}^\top \Lambda \mathbf{u}$, where $\mathbf{u} = Q^\top \mathbf{x}$.

Theorem 4. *The type of conic section that $Ax^2 + 2Bxy + Cy^2 + Dx + Ey = F$ represents is determined as follows.*

(1) *Write $M = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$.*
(2) *Find the eigenvalues λ_1 and λ_2 of M.*
(3) *Determine the conic section.*
   - *If λ_1 and λ_2 are both positive or both negative, then it is an ellipse (possibly degenerated).*
   - *If one of λ_1 and λ_2 is positive and the other is negative, then it is a hyperbola (possibly degenerated).*
   - *If one of λ_1 or λ_2 is 0, then it is a parabola.*

In $\mathbb{R}^3$, the most general form of a quadratic equation is given by

$$Ax^2 + By^2 + Cz^2 + 2Dxy + 2Exz + 2Fyz + Gx + Hy + Iz = J.$$

There are five major types of quadric surfaces.

- Ellipsoid: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = J$.
  - If $J = 1$, or $J > 0$, then it is nondegenerated.
  - if $J = 0$, then it is a single point.
  - If $J < 0$, then it is empty.
- Hyperboloid: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = J$.
  - If $J = 1$, or $J > 0$, then it is nondegenerated with one sheet.
  - if $J = 0$, then it is degenerated into an elliptic cone.
  - If $J = -1$, or $J < 0$, then it is nondegenerated with top and bottom sheets.
- Elliptic paraboloid: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{z}{c}$.
  - If $c = 1$, or $c > 0$, then it is nondegenerated opening upward.
  - If $c = -1$, or $c < 0$, then it is nondegenerated opening downward.
- Hyperbolic paraboloid: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = \dfrac{z}{c}$.
  - If $c = 1$, or $c > 0$, then it is nondegenerated with a "smiling" parabola on the xz-plane.
  - If $c = -1$, or $c < 0$, then it is nondegenerated with a "crying" parabola on the xz-plane.
- Cylinders: If there is a variable missing from the equation, then it is a "cylinder".

Theorem 5. *The type of quadric surface that $Ax^2 + By^2 + Cz^2 + 2Dxy + 2Exz + 2Fyz + Gx + Hy + Iz = J$ represents is determined as follows.*

(1) *Write* $M = \begin{pmatrix} A & D & E \\ D & B & F \\ E & F & C \end{pmatrix}$.

(2) *Find the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of M.*

(3) *Determine the conic section.*
   - *If $\lambda_1, \lambda_2, \lambda_3$ are all positive or all negative, then it is an ellipsoid (possibly degenerated).*
   - *If two of $\lambda_1, \lambda_2, \lambda_3$ are of one sign, and the other is of the opposite sign, then it is a hyperboloid (possibly degenerated).*
   - *If one of $\lambda_1, \lambda_2, \lambda_3$ is 0, and the other two are of the same sign, then it is an elliptic paraboloid (possibly degenerated into an elliptic cylinder).*
   - *If one of $\lambda_1, \lambda_2, \lambda_3$ is 0, and the other two are of opposite signs, then it is a hyperbolic paraboloid (possibly degenerated into an hyperbolic cylinder).*
   - *If two of $\lambda_1, \lambda_2, \lambda_3$ are 0, then it is a parabolic cylinder.*

13.1 — Curves in space and their tangents

Let I be an interval in $\mathbb{R}$. A **curve** in $\mathbb{R}^m$ is a **vector-valued function** $\mathbf{x} : I \to \mathbb{R}^m$ such that

$$\mathbf{x}(t) = \big(x_1(t), x_2(t), \ldots, x_m(t)\big).$$

Each x_i is a component function of $\mathbf{x}$, and it is **scalar-valued**, i.e. the conventional type.

Example 6. Graph $\mathbf{x}(t) = (\cos t, \sin t)$, $t \in \mathbb{R}$.

Solution. Since $x = \cos t$, $y = \sin t$, we have $x^2 + y^2 = 1$. In other words, the curve will be a unit circle centered at the origin.

□

Example 7. Graph $\mathbf{x}(t) = (\cos t, \sin t, t)$, $t \in \mathbb{R}$.

Example 8. Graph $\mathbf{x}(t) = (1, t, t^2)$, $t \in [-1, 1]$.

Example 9 (Cycloid)**.** Let C be the unit circle $x^2 + (y-1)^2 = 1$, and let $\mathbf{p}$ be a point on C, initially at the origin. When the circle C rotates tangentially on the x-axis (without slipping), the point $\mathbf{p}$ makes a trace in $\mathbb{R}^2$. Write the curve thus produced as a vector-valued function.

Solution. Let $(t, 1)$ be the position of the center of C at time t. The position of $\mathbf{p}$ at time t is given by

$$\mathbf{x}(t) = (t, 1) + (-\sin t, -\cos t) = (t - \sin t, 1 - \cos t).$$

□

Warning: A curve is a geometric object with "parametrization". For example, $\mathbf{x}(t) = (\sin t, \cos t)$, $t \in \mathbb{R}$ is also a unit circle centered at the origin, but unlike in Example 7, the parametrization has a different initial position at $t = 0$, and it goes clockwise instead of counterclockwise. Therefore, we treat them as different curves.

A curve $\mathbf{x} : [a, b] \to \mathbb{R}^m$ is **closed** if $\mathbf{x}(a) = \mathbf{x}(b)$, and it is **simple** (not self-intersecting) if $\mathbf{x}(t_1) \neq \mathbf{x}(t_2)$ for all $t_1 \neq t_2$ unless $\{t_1, t_2\} = \{a, b\}$.

Analytic Properties

Definition 10. Let $\mathbf{x} : I \to \mathbb{R}^m$ be a curve. Let $t_0 \in I$ and $L \in \mathbb{R}^m$. Then

$$\lim_{t \to t_0} \mathbf{x}(t) = L$$

if for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\text{for all } t \in I \text{ satisfying } 0 < |t - t_0| < \delta,\ \|\mathbf{x}(t) - L\| < \epsilon.$$

It turns out that

$$\lim_{t \to t_0} \mathbf{x}(t) = \left(\lim_{t \to t_0} x_1(t), \lim_{t \to t_0} x_2(t), \ldots, \lim_{t \to t_0} x_m(t) \right),$$

provided that the limit on either side exists.

A curve is **continuous at** t_0 if

$$\lim_{t \to t_0} \mathbf{x}(t) = \mathbf{x}(t_0).$$

If a curve is continuous at every t_0 in I, then the curve is **continuous**.

Definition 11. Let $\mathbf{x} : I \to \mathbb{R}^m$ be a curve. Let $t_0 \in I$. Then the **derivative** of $\mathbf{x}(t)$ at t_0 is defined to be

$$\mathbf{x}'(t_0) = \left. \frac{d\mathbf{x}(t)}{dt} \right|_{t=t_0} = \lim_{t \to t_0} \frac{\mathbf{x}(t) - \mathbf{x}(t_0)}{t - t_0}$$

if it exists. Furthermore, if $\mathbf{x}'(t_0)$ exists, then $\mathbf{x}(t)$ is said to be **differentiable** at t_0.

It turns out that
$$\mathbf{x}'(t_0) = \big(x_1'(t_0), x_2'(t_0), \dots, x_m'(t_0)\big),$$
provided that either side exists.

Here are some other terminologies.

- The derivative $\mathbf{x}'(t_0)$ is also known as the **tangent vector** or **velocity vector**.
- The parametric form of the **tangent line** at t_0 is $\{\mathbf{x}(t_0) + t\mathbf{x}'(t_0) : t \in \mathbb{R}\}$.
- The **speed** at t_0 is $\|\mathbf{x}'(t_0)\|$. (**Warning**: $\neq \left.\dfrac{d\|\mathbf{x}(t)\|}{dt}\right|_{t=t_0}$)
- The **direction of motion** at t_0 is $\dfrac{\mathbf{x}'(t_0)}{\|\mathbf{x}'(t_0)\|}$.
- The **acceleration vector** at t_0 is $\mathbf{x}''(t_0)$.

If a curve $\mathbf{x}(t)$ is differentiable at every t_0 in I, then the curve $\mathbf{x}(t)$ is **differentiable**. Furthermore, if $\mathbf{x}'(t)$ is continuous and never $\mathbf{0}$, then the curve $\mathbf{x}(t)$ is **smooth**. If a curve $\mathbf{x}(t)$ is continuous, and I can be partitioned into finitely many subintervals such that $\mathbf{x}(t)$ is smooth on each subinterval, then the curve $\mathbf{x}(t)$ is **piecewise smooth**.

Here are some rules for differentiation of vector-valued functions.

Let $\mathbf{x}, \mathbf{y} : I \to \mathbb{R}^m$ be two differentiable curves. Let $\mathbf{C}$ be a constant vector, c be a scalar, and $f : I \to \mathbb{R}$ be a differentiable scalar-valued function.

- (constant function rule) $\mathbf{C}' = \mathbf{0}$.
- (sum and difference rule) $[\mathbf{x}(t) \pm \mathbf{y}(t)]' = \mathbf{x}'(t) + \mathbf{y}'(t)$.
- (product rules)
  - $[c\mathbf{x}(t)]' = c\mathbf{x}'(t)$.
  - $[f(t)\mathbf{x}(t)]' = f'(t)\mathbf{x}(t) + f(t)\mathbf{x}'(t)$.
  - $[\mathbf{x}(t) \cdot \mathbf{y}(t)]' = \mathbf{x}'(t) \cdot \mathbf{y}(t) + \mathbf{x}(t) \cdot \mathbf{y}'(t)$.
  - If $m = 3$, $[\mathbf{x}(t) \times \mathbf{y}(t)]' = \mathbf{x}'(t) \times \mathbf{y}(t) + \mathbf{x}(t) \times \mathbf{y}'(t)$.
- (chain rule) $[\mathbf{x}(f(t))]' = f'(t)\mathbf{x}'(f(t))$.

Example 12. Let $\mathbb{S}^{m-1}$ denote the $(m-1)$-dimensional unit sphere centered at the origin in $\mathbb{R}^m$. Let $\mathbf{x} : I \to \mathbb{S}^{m-1}$ be a curve on the sphere. Show that the position vector $\mathbf{x}(t)$ and the tangent vector $\mathbf{x}'(t)$ are orthogonal for all $t \in I$.

Proof.
$$\begin{aligned}
\|\mathbf{x}(t)\| &= 1 \\
\mathbf{x}(t) \cdot \mathbf{x}(t) &= 1 \\
[\mathbf{x}(t) \cdot \mathbf{x}(t)]' &= 0 \\
\mathbf{x}'(t) \cdot \mathbf{x}(t) + \mathbf{x}(t) \cdot \mathbf{x}'(t) &= 0 \\
2\mathbf{x}(t) \cdot \mathbf{x}'(t) &= 0 \\
\mathbf{x}(t) \cdot \mathbf{x}'(t) &= 0
\end{aligned}$$
This implies that $\mathbf{x}'(t)$ and $\mathbf{x}(t)$ are orthogonal.

□