MATH 2010E ADVANCED CALCULUS I
LECTURE 2

WING HONG TONY WONG

12.4 — Cross product

Given $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$, the **cross product** is given by

$$\mathbf{x} \times \mathbf{y} = \begin{vmatrix} \widehat{\mathbf{i}} & \widehat{\mathbf{j}} & \widehat{\mathbf{k}} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix},$$

or

$$\mathbf{x} \times \mathbf{y} = \left(\begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix}, - \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix}, \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \right).$$

Warnings:

(1) Remember the negative sign in the second term.
(2) Unlike dot product, **cross product** is only defined for vectors in $\mathbb{R}^3$. (If we want to define something analogous in $\mathbb{R}^n$, we need to take the "product" of $n-1$ vectors.)
(3) The cross product of two vectors in $\mathbb{R}^3$ produces a vector in $\mathbb{R}^3$, instead of a real number.

Example 1. If $\mathbf{x} = (1, 1, 2)$ and $\mathbf{y} = (-1, 2, 1)$, then

$$\mathbf{x} \times \mathbf{y} = \left(\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}, - \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} \right) = (-3, -3, 3).$$

If $\mathbf{x} = \mathbf{j}$ and $\mathbf{y} = \mathbf{i}$, then

$$\mathbf{j} \times \mathbf{i} = \left(\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}, - \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \right) = (0, 0, -1) = -\mathbf{k}.$$

The geometric interpretation of cross product is given by the following theorem.

Theorem 2. *If $\mathbf{x}$ and $\mathbf{y}$ are not scalar multiples of each other, then $\mathbf{x} \times \mathbf{y}$ is the unique vector in $\mathbb{R}^3$ such that*

(*a*) $\mathbf{x} \times \mathbf{y}$ *is orthogonal to both* $\mathbf{x}$ *and* $\mathbf{y}$.
(*b*) *The orientation of* $\{\mathbf{x}, \mathbf{y}, \mathbf{x} \times \mathbf{y}\}$ *is "right-handed".*
(*c*) $\|\mathbf{x} \times \mathbf{y}\| = \|\mathbf{x}\| \|\mathbf{y}\| |\sin\theta|$ *is the area of the parallelogram spanned by* $\mathbf{x}$ *and* $\mathbf{y}$.

Lemma 3 (scalar triple product)**.** *Let $\mathbf{x}, \mathbf{y}, \mathbf{z}$ be three vectors in $\mathbb{R}^3$. Then*

$$(\mathbf{x} \times \mathbf{y}) \cdot \mathbf{z} = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}.$$

Date: Thursday, 16^{th} May, 2015.

Proof.

$$\begin{aligned}(\mathbf{x}\times\mathbf{y})\cdot\mathbf{z} &= \left(\begin{vmatrix} x_2 & x_3 \\ y_2 & y_3\end{vmatrix}, \begin{vmatrix} x_3 & x_1 \\ y_3 & y_1\end{vmatrix}, \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2\end{vmatrix}\right)\cdot(z_1, z_2, z_3) \\ &= z_1\begin{vmatrix} x_2 & x_3 \\ y_2 & y_3\end{vmatrix} - z_2\begin{vmatrix} x_1 & x_3 \\ y_1 & y_3\end{vmatrix} + z_3\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2\end{vmatrix} \\ &= \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3\end{vmatrix}\end{aligned}$$

by the expansion of the last row in the 3×3 determinant.

□

Proof of Theorem 2. (a) By Lemma 3,

$$(\mathbf{x}\times\mathbf{y})\cdot\mathbf{x} = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ x_1 & x_2 & x_3\end{vmatrix} = 0.$$

Similar proof works for $(\mathbf{x}\times\mathbf{y})\cdot\mathbf{y}$.
(c)

$$\begin{aligned}\|\mathbf{x}\times\mathbf{y}\|^2 = &\begin{vmatrix} x_2 & x_3 \\ y_2 & y_3\end{vmatrix}^2 + \left(-\begin{vmatrix} x_1 & x_3 \\ y_1 & y_3\end{vmatrix}\right)^2 + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2\end{vmatrix}^2 \\ =&x_1^2y_2^2 + x_1^2y_3^2 + x_2^2y_1^2 + x_2^2y_3^2 + x_3^2y_1^2 + x_3^2y_2^2 - 2x_1x_2y_1y_2 - 2x_2x_3y_2y_3 - 2x_1x_3y_1y_3 \\ =&(x_1^2+x_2^2+x_3^2)(y_1^2+y_2^2+y_3^2) - (x_1y_1+x_2y_2+x_3y_3)^2 \\ =&\|\mathbf{x}\|^2\|\mathbf{y}\|^2 - (\mathbf{x}\cdot\mathbf{y})^2 \\ =&\|\mathbf{x}\|^2\|\mathbf{y}\|^2(1-\cos^2\theta) = \|\mathbf{x}\|^2\|\mathbf{y}\|^2\sin^2\theta.\end{aligned}$$

(b) Recall that the orientation of the basis is decided by the sign of the determinant. By Lemma 3,

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3\end{vmatrix} & -\begin{vmatrix} x_1 & x_3 \\ y_1 & y_3\end{vmatrix} & \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2\end{vmatrix}\end{vmatrix} = (\mathbf{x}\times\mathbf{y})\cdot(\mathbf{x}\times\mathbf{y}) = \|\mathbf{x}\times\mathbf{y}\|^2.$$

As $\mathbf{x}$ and $\mathbf{y}$ are not scalar multiples of each other, $\mathbf{x}$ and $\mathbf{y}$ are not parallel, i.e. $\theta\neq 0$ or π. Hence, $\|\mathbf{x}\times\mathbf{y}\|^2 = \|\mathbf{x}\|^2\|\mathbf{y}\|^2\sin^2\theta > 0$.

□

The geometric interpretation of scalar triple product is given by the following theorem.

Theorem 4. *Let $\mathbf{x}, \mathbf{y}, \mathbf{z}$ be three vectors in $\mathbb{R}^3$. If $\mathbf{x}$ and $\mathbf{y}$ are not scalar multiples of each other, then $(\mathbf{x}\times\mathbf{y})\cdot\mathbf{z}$ is the (signed) volume of the parallelopipe spanned by $\mathbf{x}, \mathbf{y}, \mathbf{z}$.*

Proof. $(\mathbf{x}\times\mathbf{y})\cdot\mathbf{z} = \|\mathbf{x}\times\mathbf{y}\|\|\mathbf{z}\|\cos\phi$, where ϕ is the angle between $\mathbf{x}\times\mathbf{y}$ and $\mathbf{z}$. By Theorem 2(c), we know that $\|\mathbf{x}\times\mathbf{y}\|$ is the base area of the parallelopipe. By Theorem 2(a), $\mathbf{x}\times\mathbf{y}$ is orthogonal to the base, so $\|\mathbf{z}\|\cos\phi$ is the (signed) height of the parallelopipe.

□

Corollary 5.

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$$

is the (signed) volume of the parallelopipe spanned by (x_1, x_2, x_3), (y_1, y_2, y_3), *and* (z_1, z_2, z_3).

Corollary 6. *The nonzero vectors* $\mathbf{x}$ *and* $\mathbf{y}$ *in* $\mathbb{R}^3$ *are parallel to each other if and only if* $\mathbf{x} \times \mathbf{y} = \mathbf{0}$.

Here is a list of properties of cross product.

(1) (anti-commutative) $\mathbf{x} \times \mathbf{y} = -\mathbf{y} \times \mathbf{x}$.
(2) (distributive) $(\mathbf{x} + \mathbf{y}) \times \mathbf{z} = \mathbf{x} \times \mathbf{z} + \mathbf{y} \times \mathbf{z}$ and $\mathbf{x} \times (\mathbf{y} + \mathbf{z}) = \mathbf{x} \times \mathbf{y} + \mathbf{x} \times \mathbf{z}$.
(3) (commutative and associative with scalar multiplication)
$(\lambda_1 \mathbf{x}) \times (\lambda_2 \mathbf{y}) = (\lambda_1 \lambda_2)(\mathbf{x} \times \mathbf{y})$.
(4) (vector triple product) $\mathbf{x} \times (\mathbf{y} \times \mathbf{z}) = (\mathbf{x} \cdot \mathbf{z})\mathbf{y} - (\mathbf{x} \cdot \mathbf{y})\mathbf{z}$.

Warning: From property (4), we know that cross product is not associative.

From property (4), we also obtain the following identity.

$$\mathbf{x} \times (\mathbf{y} \times \mathbf{z}) + \mathbf{y} \times (\mathbf{z} \times \mathbf{x}) + \mathbf{z} \times (\mathbf{x} \times \mathbf{y}) = \mathbf{0}.$$

12.5 — Lines and planes in $\mathbb{R}^3$

As mentioned in Section 12.1, we need a system of two equations to describe a (straight) line in $\mathbb{R}^3$, since one equation gives a 2-dimensional plane, and two equations give the intersection of two planes, i.e. a line.

Here, we introduce another method to express a line algebraically. A line can be determined by passing through a point $\mathbf{p} = (x_0, y_0, z_0)$ and moving along a specific direction $\mathbf{v} = (v_1, v_2, v_3)$. Hence, we obtain a **parametric form** of a line

$$\{\mathbf{p} + t\mathbf{v} : t \in \mathbb{R}\},$$

or

$$\{(x_0 + tv_1, y_0 + tv_2, z_0 + tv_3) : t \in \mathbb{R}\},$$

where t is called the **free parameter**.

In **parametric equations form**,

$$\begin{aligned} x &= x_0 + tv_1 \\ y &= y_0 + tv_2 \\ z &= z_0 + tv_3. \end{aligned}$$

If we make t the subject in every equation, we get

$$\frac{x - x_0}{v_1} = \frac{y - y_0}{v_2} = \frac{z - z_0}{v_3},$$

which is called the **symmetric form**.

Example 7. Let L be the line defined by

$$\begin{aligned} x - 2y + z &= 1 \\ 2x + y - z &= 0. \end{aligned}$$

Find the parametric form of L.

Solution. Label the first equation by (1) and the second equation by (2). Then (2)−(1)×2 yields

$$5y - 3z = -2,$$

or

$$y = \frac{3z - 2}{5}.$$

Let $z = t$ be a free parameter. Then $y = \dfrac{3t-2}{5}$, and $x - 2\left(\dfrac{3t-2}{5}\right) + t = 1$ implies $x = \dfrac{t+1}{5}$. Hence, the parametric form of L is

$$\left\{ \left(\frac{t+1}{5}, \frac{3t-2}{5}, t \right) : t \in \mathbb{R} \right\},$$

or

$$\left\{ \left(\frac{1}{5}, -\frac{2}{5}, 0 \right) + t \left(\frac{1}{5}, \frac{3}{5}, 1 \right) : t \in \mathbb{R} \right\}.$$

□

The parametric form of a line has the benefit that it carries all important physical information in one formula. More specifically, in

$$\mathbf{p} + t\mathbf{v} = \mathbf{p} + t\|\mathbf{v}\| \frac{\mathbf{v}}{\|\mathbf{v}\|},$$

$\mathbf{p}$ represents the initial position, t denotes the time, $\|\mathbf{v}\|$ stands for the speed, and $\dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ gives the direction of the motion.

Example 8. Consider $\mathbb{R}^3$. Find the perpendicular distance from the point $\mathbf{x}$ to the line $\mathbf{p} + t\mathbf{v}$.

Solution. Let θ be the angle between $\mathbf{x} - \mathbf{p}$ and $\mathbf{v}$. The desired perpendicular distance is

$$\begin{aligned} \|\mathbf{x} - \mathbf{p}\| |\sin\theta| &= \frac{\|\mathbf{x} - \mathbf{p}\| \|\mathbf{v}\| |\sin\theta|}{\|\mathbf{v}\|} \\ &= \left\| (\mathbf{x} - \mathbf{p}) \times \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\|. \end{aligned}$$

□

As mentioned in Section 12.3, a plane can be determined by passing through a point $\mathbf{p} = (x_0, y_0, z_0)$ and have a normal vector $\mathbf{n} = A\widehat{\mathbf{i}} + B\widehat{\mathbf{j}} + C\widehat{\mathbf{k}}$. Any point $\mathbf{x}$ on the plane has the property that

$$\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \big((\mathbf{x} - \mathbf{p}) + \mathbf{p} \big) = \mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) + \mathbf{n} \cdot \mathbf{p} = \mathbf{n} \cdot \mathbf{p}$$

since $\mathbf{x} - \mathbf{p}$ lies on the plane and is orthogonal to $\mathbf{n}$, i.e. $\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$. Therefore, the plane can be described by

$$\{\mathbf{x} \in \mathbb{R}^3 : \mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}\},$$

or

$$\{(x,y,z) \in \mathbb{R}^3 : Ax + By + Cz = Ax_0 + By_0 + Cz_0\}.$$

An equation of the plane is commonly written as

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

or

$$Ax + By + Cz = D,$$

where $D = Ax_0 + By_0 + Cz_0$.

Example 9. Consider $\mathbb{R}^3$. Find the perpendicular distance form the point $\mathbf{x}$ to the plane

$$Ax + By + Cz = D.$$

Solution. First, we need to find an arbitrary point $\mathbf{p}$ on the plane. In order to do so, note that A, B, and C cannot be all zero. For the easiness of demonstration in this problem, let us assume that $B \neq 0$. Then by taking $x = z = 0$, we solve that $y = \frac{D}{B}$. In other words, $\mathbf{p} = \left(0, \frac{D}{B}, 0\right)$.

Let θ be the angle between $\mathbf{x} - \mathbf{p}$ and the normal of the plane $\mathbf{n} = (A, B, C)$. The desired perpendicular distance is

$$\begin{aligned} \|\mathbf{x} - \mathbf{p}\| |\cos\theta| &= \frac{\|\mathbf{x} - \mathbf{p}\| \|\mathbf{n}\| |\cos\theta|}{\|\mathbf{n}\|} \\ &= \left| (\mathbf{x} - \mathbf{p}) \cdot \frac{\mathbf{n}}{\|\mathbf{n}\|} \right|. \end{aligned}$$

□

Example 10. Find an equation of the plane that passes through $(1,2,0)$, $(0,-1,3)$, and $(-2,1,1)$.

Solution. A normal to the plane is

$$\begin{aligned} \mathbf{n} &= \big((0,-1,3) - (1,2,0)\big) \times \big((-2,1,1) - (1,2,0)\big) \\ &= \begin{vmatrix} \widehat{\mathbf{i}} & \widehat{\mathbf{j}} & \widehat{\mathbf{k}} \\ -1 & -3 & 3 \\ -3 & -1 & 1 \end{vmatrix} \\ &= (0, -8, -8). \end{aligned}$$

Hence, the plane is

$$\{(x,y,z) \in \mathbb{R}^3 : (0,-8,-8) \cdot (x,y,z) = (0,-8,-8) \cdot (1,2,0)\}.$$

Therefore, the equation of the plane is

$$-8y - 8z = -16,$$

or

$$y + z = 2.$$

(You may also solve this problem by letting the equation of the plane as

$$Ax + By + C = D$$

and setting up three equations through plugging the three points into the equation.)

□

Example 11. Find a vector parallel to the line of intersection of the planes

$$4x + y - 3z = 6$$
$$-2x + 4y - z = 1.$$

Solution. Note that the line of intersection is orthogonal to the normals of the two planes. Hence, a desired vector is

$$(4, 1, -3) \times (-2, 4, -1) = \begin{vmatrix} \widehat{\mathbf{i}} & \widehat{\mathbf{j}} & \widehat{\mathbf{k}} \\ 4 & 1 & -3 \\ -2 & 4 & -1 \end{vmatrix} = (11, 10, 18).$$

(You may also solve this problem by finding the parametric form $\mathbf{p} + t\mathbf{v}$ of the line of intersection using the method in Example 7. Then $\mathbf{v}$ is the desired answer.)

$\square$

Example 12. Find the angle between the two planes

$$4x + y - 3z = 6$$

and

$$-2x + 4y - z = 1.$$

Solution. Note that the angle between the two planes is the same as the angle between the normals of the two planes. Hence, the angle is

$$\theta = \cos^{-1}\left(\frac{(4, 1, -3) \cdot (-2, 4, -1)}{\|(4, 1, -3)\|\|(-2, 4, -1)\|}\right) = \cos^{-1}\left(\frac{-1}{\sqrt{546}}\right) \approx 1.6136 \quad \text{(in radian measure)}.$$

(Notice that you may also answer with $\pi - \cos^{-1}\left(\frac{-1}{\sqrt{546}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{546}}\right)$.)

$\square$

We can also write a plane in **parametric form**

$$\{\mathbf{p} + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 : t_1, t_2 \in \mathbb{R}\}.$$

There are two free parameters since there are two degrees of freedom on a plane.

Example 13. Let P be the plane defined by

$$x - 2y + 3z = 4.$$

Find the parametric form of P.

Solution. Let $y = t_1$ and $z = t_2$ be two free parameters. Then $x = 2t_1 - 3t_2 + 4$. Hence, the parametric form of P is

$$\{(2t_1 - 3t_2 + 4, t_1, t_2) : t_1, t_2 \in \mathbb{R}\},$$

or

$$\{(4, 0, 0) + t_1(2, 1, 0) + t_2(-3, 0, 1) : t_1, t_2 \in \mathbb{R}\}.$$

$\square$