MATH 2010E ADVANCED CALCULUS I
LECTURE 1

WING HONG TONY WONG

Course information

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Assessment scheme

HW	15%	
Mid-term	35%	Tuesday, 2^{nd} June, 2015 (Tentative)
Final	50%	Thursday, 25^{th} June, 2015

Outline of the course

In the past, we have dealt with calculus about functions on a single variable, i.e. $f : \mathbb{R} \to \mathbb{R}$. Examples of such functions include

- Polynomials: $f(x) = 4x^3 + \frac{8}{3}x - \sqrt{5}$;
- Trigonometric functions: $f(x) = \sin(x)$, $f(x) = \cos(x^3)$;
- Exponentials and Logarithms: $f(x) = e^x$, $f(x) = 3^{4x+2}$, $f(x) = \log_6(x)$.

Calculus has two branches: differentiation and integration.

Differentiation

- $\dfrac{df}{dx}$, $f'(x)$.
- Tangent lines, slopes.
- Product rule, Quotient rule, Chain rule.
- Taylor's theorem.
- Implicit differentiation.
- Finding maximum/minimum of $f(x)$.

Integration

- $\displaystyle\int f(x)dx$, $\displaystyle\int_a^b f(x)dx$.
- Areas under curve.
- Substitution, Integration by part.
- Finding volume of revolution.

Date: Tuesday, 12^{th} May, 2015.

The relation between differentiation and integration is given by the following.

Fundamental Theorem(s) of Calculus

(1) $\displaystyle\int_a^b f'(x)dx = f(b) - f(a)$.

(2) $\displaystyle\frac{d}{dx}\int_a^x f(t)dt = f(x)$.

In Advanced Calculus, we deal with calculus about functions on multivariables, i.e. $F : \mathbb{R}^n \to \mathbb{R}^m$, and

$$F(x_1, \ldots, x_n) = \big(f_1(x_1, \ldots, x_n),\ \ldots,\ f_m(x_1, \ldots, x_n)\big).$$

MATH 2010 deals with

- Differentiation;
- Related problems such as maximization and minimization of $F(x_1, \ldots, x_n)$ subject to constraints.

MATH 2020 deals with

- Integration;
- Related problems such as fundamental theorem of calculus.

Acknowledgement: All the lecture notes in this course will follow very closely to those by Dr. Martin Li.

12.1 — Three-dimensional coordinate systems

A real line is a one-dimensional object, denoted by $\mathbb{R}$. Every point on the line is a real number x.

A Cartesian coordinate plane, or a rectangular coordinate plane, is a two-dimensional coordinate system. Every point on the plane is denoted by coordinates

$$(x, y),$$

where x and y are real numbers. The Cartesian coordinate plane is often denoted by $\mathbb{R} \times \mathbb{R}$, or $\mathbb{R}^2$. In set notation,

$$\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}.$$

Question 1.

(a) What does the equation $x^2 + y^2 - 4y - 2 = 0$ correspond to in $\mathbb{R}^2$?
(b) What does the inequality $3x - 2y > 8$ correspond to in $\mathbb{R}^2$?
(c) What does the inequality $y \leq x^2$ correspond to in $\mathbb{R}^2$?
(d) Why is there a correspondence between an algebraic equation or inequality and a geometric figure in $\mathbb{R}^2$?

Question 2.

(a) What is the equation of the x-axis in $\mathbb{R}^2$?
(b) What does the equation $x = -3$ correspond to in $\mathbb{R}^2$?

A three-dimensional Cartesian coordinate space is denoted by $\mathbb{R}^3$. Every point on the plane is denoted by coordinates

$$(x, y, z),$$

where x, y, and z are real numbers. In set notation,

$$\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}.$$

Question 3.
(a) How are the three axes in $\mathbb{R}^3$ arranged?
(b) What is the equation of the x-axis in $\mathbb{R}^3$? (Beware of the trap!)
(c) What does the equation $x = -3$ correspond to in $\mathbb{R}^3$?

In $\mathbb{R}^3$, the equations $x = 0$, $y = 0$, and $z = 0$ correspond to the yz-plane, xz-plane, and xy-plane respectively. The x-axis is the intersection of the xy-plane and xz-plane, so it is described by a system of linear equations

$$\begin{cases} y = 0 \\ z = 0 \end{cases}.$$

Roughly speaking, each equation is a restriction to the dimension of freedom, hence in $\mathbb{R}^3$, one equation corresponds to a two-dimensional surface, and a system of two equations usually corresponds to a one-dimensional curve.

Example 4. Interpret the following equations and inequalities geometrically.

(a)	$z \geq 2$	**half-space** on or above the plane $z = 2$.
(b)	$z = 0,\ x \leq 0,\ y \geq 0$	second quadrant of the xy-plane.
(c)	$x \geq 0,\ y \geq 0,\ z \geq 0$	first **octant**.
(d)	$-1 \leq y \leq 1$	**slab** between the planes $y = -1$ and $y = 1$ (planes included).
(e)	$y = -2,\ z = 2$	straight line parallel to x-axis through the point $(0, -2, 2)$.

Theorem 5. *Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be two points in $\mathbb{R}^3$. Then the distance between them is*

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

The equation of a **sphere** with radius r and center (x_0, y_0, z_0) is

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2.$$

Example 6. Interpret the following equations and inequalities geometrically.

(a)	$(x-1)^2 + (y+2)^2 + (z-3)^2 < 16$	**interior** of the ball with radius 4 and center $(1, -2, 3)$.
(b)	$(x-1)^2 + (y+2)^2 + (z-3)^2 \leq 16$	**solid ball** with radius 4 and center $(1, -2, 3)$.
(c)	$(x-1)^2 + (y+2)^2 + (z-3)^2 > 16$	**exterior** of the ball with radius 4 and center $(1, -2, 3)$.
(d)	$(x-1)^2 + (y+2)^2 + (z-3)^2 = 16,\ z \leq 0$	**lower hemisphere** with radius 4 and center $(1, -2, 3)$ (**equator** included).

Exercise 7. Interpret the following equations and inequalities geometrically.
(a) (12.1.14) $x^2 + y^2 + z^2 = 4$, $y = x$.
(b) (12.1.20b) $x^2 + y^2 \leq 1$, $z = 3$.
(c) (12.1.56) $x^2 + y^2 + z^2 - 6y + 8z = 0$.

Exercise 8. (12.1.26b) Describe the given set with a single equation or with a pair of equations.

The plane through the point $(3, -1, 2)$ perpendicular to the y-axis.

12.2 — Vectors

More generally, we can talk about the n-dimensional Euclidean space

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_1, x_2, \ldots, x_n \in \mathbb{R}\}.$$

A point $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$ is also referred as a **position vector**, with the tail of the vector at the origin $\mathbf{0} = (0, 0, \ldots, 0)$, and the head of the vector at the point $\mathbf{x}$. In hand-written form, we also write $\mathbf{x}$ as $\overrightarrow{\mathbf{x}}$.

A **vector** from the point $\mathbf{x}$ to the point $\mathbf{y}$ is given by $\mathbf{y} - \mathbf{x} = (y_1 - x_1, \ldots, y_n - x_n)$. The sum of two vectors $\mathbf{x} + \mathbf{y} = (x_1 + y_1, \ldots, x_n + y_n)$ can be found by the **parallelogram law**.

The **length**, or the **norm** of a vector $\mathbf{x} = (x_1, \ldots, x_n)$ is given by the Euclidean formula

$$\|\mathbf{x}\| = \sqrt{x_1^2 + \cdots + x_n^2},$$

and the **distance** between points $\mathbf{x}$ and $\mathbf{y}$ is given by the norm of the vector $\mathbf{y} - \mathbf{x}$, i.e.

$$\|\mathbf{y} - \mathbf{x}\| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}.$$

In this course, a **scalar** is simply a real number. The relationship between $\mathbf{x}$ and the **scalar multiplication** $\lambda\mathbf{x} = (\lambda x_1, \ldots, \lambda x_n)$ is given by the following table.

$\lambda > 0$	$\lambda\mathbf{x}$ and $\mathbf{x}$ are in the same direction
$\lambda < 0$	$\lambda\mathbf{x}$ and $\mathbf{x}$ are in opposite directions
$\lvert\lambda\rvert > 1$	$\lambda\mathbf{x}$ is a stretching of $\mathbf{x}$
$\lvert\lambda\rvert < 1$	$\lambda\mathbf{x}$ is a shrinking of $\mathbf{x}$

Here is a list of properties of vector addition and scalar multiplication.

(1) (commutative) $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$.
(2) (associative) $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$.
(3) (existence of additive identity) $\mathbf{x} + \mathbf{0} = \mathbf{x}$.
(4) (existence of additive inverse) $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$.
(5) (distributive) $\lambda(\mathbf{x} + \mathbf{y}) = \lambda\mathbf{x} + \lambda\mathbf{y}$.
(6) (distributive) $(\lambda_1 + \lambda_2)\mathbf{x} = \lambda_1\mathbf{x} + \lambda_2\mathbf{x}$.
(7) (associative with scalar multiplication) $\lambda_1(\lambda_2\mathbf{x}) = (\lambda_1\lambda_2)\mathbf{x}$.
(8) (fixed by scalar multiplication of 1) $1\mathbf{x} = \mathbf{x}$.

Hence, $\mathbb{R}^n$ with addition and scalar multiplication defined forms a **vector space** (introduced in MATH 1030 — Linear Algebra).

A set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ forms a **basis** of $\mathbb{R}^n$ if every $\mathbf{u}$ in $\mathbb{R}^n$ can be written uniquely as

$$\mathbf{u} = k_1\mathbf{v}_1 + \cdots k_n\mathbf{v}_n.$$

The standard basis of $\mathbb{R}^n$ is $\big\{\mathbf{e}_1 = (1, 0, \dots, 0), \mathbf{e}_2 = (0, 1, 0, \dots, 0), \dots, \mathbf{e}_n = (0, \dots, 0, 1)\big\}$.

Question 9. We know that $\mathbf{e}_1, \dots, \mathbf{e}_n$ are mutually orthogonal in $\mathbb{R}^n$. How many different ways are there to arrange them? (We consider two ways the same if one can be obtained from the other by rotation.)

It turns out that the answer to Question 9 is a generalization to that of Question 3(a). There is a general technique to decide between "left-handed" orientation and "right-handed" orientation, and the method is called the **determinant** of a **matrix**.

For a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the determinant is given by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc;$$

for a 3×3 matrix $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, the determinant is given by the "arrow method"

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - (ceg + bdi + afh).$$

Warning: For 4×4 or larger matrices, the determinant CANNOT be found by using the "arrow method".

The orientation is "right-handed" if the determinant is positive, and the orientation is "left-handed" if the determinant is negative.

A vector $\mathbf{v}$ is a **unit vector** if the norm is 1. In this situation, we also write $\mathbf{v}$ as $\widehat{\mathbf{v}}$. For every vector $\mathbf{v}$,

$$\widehat{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|}.$$

The unit vector $\widehat{\mathbf{v}}$ records the **direction** of $\mathbf{v}$, while the norm $\|\mathbf{v}\|$ records the length of $\mathbf{v}$. In other words,

$$\mathbf{v} = \|\mathbf{v}\|\widehat{\mathbf{v}}.$$

Finally, the midpoint $\mathbf{m}$ between the two points $\mathbf{x}$ and $\mathbf{y}$ is given by

$$\mathbf{m} = \frac{1}{2}(\mathbf{x} + \mathbf{y}) = \left(\frac{x_1 + y_1}{2}, \dots, \frac{x_n + y_n}{2}\right).$$

12.3 — Dot product

In $\mathbb{R}^n$, the **dot product**, or the **inner product** between $\mathbf{x}$ and $\mathbf{y}$ is defined to by

$$\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n.$$

Note that the norm $\|\mathbf{x}\| = \sqrt{\mathbf{x}\cdot\mathbf{x}}$.

Here is a list of properties of dot product.

(1) (commutative) $\mathbf{x}\cdot\mathbf{y} = \mathbf{y}\cdot\mathbf{x}$.
(2) (distributive) $(\mathbf{x}+\mathbf{y})\cdot\mathbf{z} = \mathbf{x}\cdot\mathbf{z} + \mathbf{y}\cdot\mathbf{z}$.
(3) (associative with scalar multiplication) $(\lambda\mathbf{x})\cdot\mathbf{y} = \lambda(\mathbf{x}\cdot\mathbf{y})$.
(4) (positive definite) $\mathbf{x}\cdot\mathbf{x} \geq 0$, and "=" holds if and only if $\mathbf{x} = \mathbf{0}$.

Warning: There is no such thing as $\mathbf{x}\cdot\mathbf{y}\cdot\mathbf{z}$, why?

The geometric interpretation of dot product is given by the following theorem.

Theorem 10.
$$\mathbf{x}\cdot\mathbf{y} = \|\mathbf{x}\|\|\mathbf{y}\|\cos\theta,$$
where θ is the angle between the vectors $\mathbf{x}$ and $\mathbf{y}$.

Proof. Consider the triangle bounded by vectors $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{y}-\mathbf{x}$. By cosine law,
$$\|\mathbf{y}-\mathbf{x}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\|\mathbf{x}\|\|\mathbf{y}\|\cos\theta.$$
Notice that
$$\begin{aligned}
\text{L.H.S.} &= (\mathbf{y}-\mathbf{x})\cdot(\mathbf{y}-\mathbf{x}) \\
&= \mathbf{y}\cdot(\mathbf{y}-\mathbf{x}) - \mathbf{x}\cdot(\mathbf{y}-\mathbf{x}) \qquad \text{(by distributivity)} \\
&= (\mathbf{y}-\mathbf{x})\cdot\mathbf{y} - (\mathbf{y}-\mathbf{x})\cdot\mathbf{x} \qquad \text{(by commutativity)} \\
&= \mathbf{y}\cdot\mathbf{y} - \mathbf{x}\cdot\mathbf{y} - \big(\mathbf{y}\cdot\mathbf{x} - \mathbf{x}\cdot\mathbf{x}\big) \qquad \text{(by distributivity)} \\
&= \|\mathbf{y}\|^2 - 2\mathbf{x}\cdot\mathbf{y} + \|\mathbf{x}\|^2. \qquad \text{(by commutiativity)}
\end{aligned}$$
By comparing with R.H.S., we have our desired conclusion.

$\square$

Corollary 11. *The nonzero vectors $\mathbf{x}$ and $\mathbf{y}$ are orthogonal to each other if and only if $\mathbf{x}\cdot\mathbf{y} = 0$.*

Let $\mathrm{proj}_{\mathbf{y}}\mathbf{x}$ be the projection vector of $\mathbf{x}$ onto the direction of $\mathbf{y}$. By the definition of projection,
$$\|\mathrm{proj}_{\mathbf{y}}\mathbf{x}\| = \|\mathbf{x}\|\cos\theta,$$
where θ is the angle between $\mathbf{x}$ and $\mathbf{y}$. Also, $\mathrm{proj}_{\mathbf{y}}\mathbf{x}$ takes the direction $\widehat{\mathbf{y}}$. Therefore,
$$\begin{aligned}
\mathrm{proj}_{\mathbf{y}}\mathbf{x} &= \|\mathbf{x}\|\cos\theta\frac{\mathbf{y}}{\|\mathbf{y}\|} \\
&= \|\mathbf{x}\|\|\mathbf{y}\|\cos\theta\frac{\mathbf{y}}{\|\mathbf{y}\|^2} \\
&= \left(\frac{\mathbf{x}\cdot\mathbf{y}}{\mathbf{y}\cdot\mathbf{y}}\right)\mathbf{y},
\end{aligned}$$
and
$$\|\mathrm{proj}_{\mathbf{y}}\mathbf{x}\| = \frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{y}\|} = \mathbf{x}\cdot\widehat{\mathbf{y}}.$$

Example 12. Find $\mathrm{proj}_{\mathbf{y}}\mathbf{x}$ and $\|\mathrm{proj}_{\mathbf{x}}\mathbf{y}\|$ if $\mathbf{x} = (5, 2, -6)$ and $\mathbf{y} = (3, -1, 2)$.

Here are some applications of dot product.

(1) Compute the angle between two vectors $\mathbf{x}$ and $\mathbf{y}$:
$$\cos\theta = \frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|}.$$
(2) Detect orthogonality between $\mathbf{x}$ and $\mathbf{y}$ (see Corollary 11).

(3) Define a plane containing the origin in $\mathbb{R}^3$ using the **normal vector**, i.e.
$$\{\mathbf{v}\in\mathbb{R}^3 : \mathbf{v}\cdot\mathbf{x} = 0\}.$$
(4) Find a component of a vector: $\mathrm{proj}_{\mathbf{v}}\mathbf{x} = (\mathbf{x}\cdot\widehat{\mathbf{v}})\widehat{\mathbf{v}}$.

(5) Cauchy-Schwarz inequality:
$$(x_1y_1 + \cdots + x_ny_n)^2 \le (x_1^2 + \cdots + x_n^2)(y_1^2 + \cdots + y_n^2).$$
Proof. L.H.S. $= \mathbf{x}\cdot\mathbf{y} = (\|\mathbf{x}\|\|\mathbf{y}\|\cos\theta)^2 = \|\mathbf{x}\|^2\|\mathbf{y}\|^2\cos^2\theta \le \|\mathbf{x}\|^2\|\mathbf{y}\|^2 =$ R.H.S. $\square$

Example 13.

12.1 26b, 38, 59.
12.3 22.