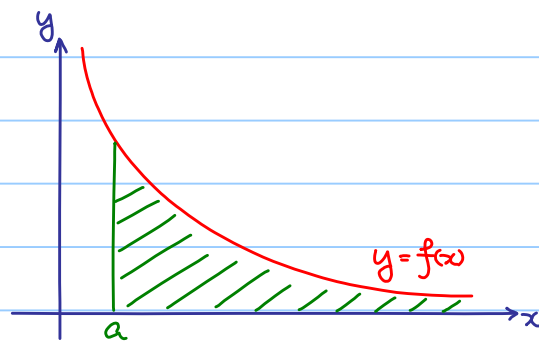
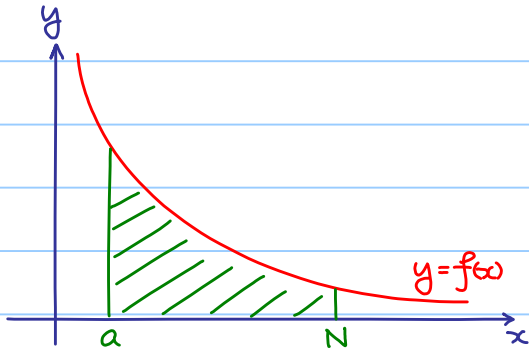


Improper Integrals :



Question: Find the area of the unbounded region?

Idea:



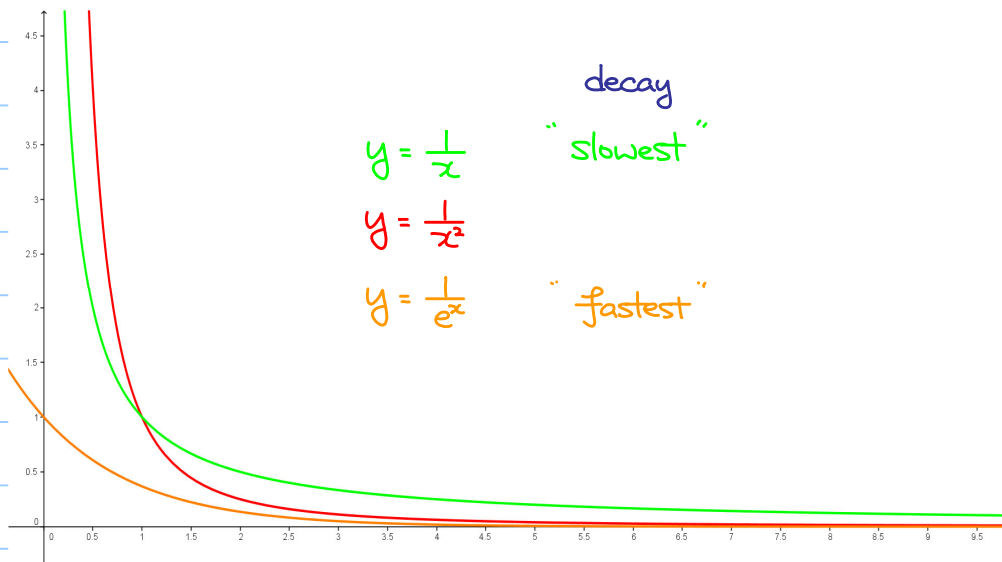
$$\int_a^N f(x) dx$$

\rightsquigarrow

Area of the unbounded region
 $= \lim_{N \rightarrow +\infty} \int_a^N f(x) dx$ (if it exists)

We denote it by $\int_a^{+\infty} f(x) dx$

e.g.



$$\textcircled{1} \lim_{N \rightarrow +\infty} \int_1^N \frac{1}{x} dx = \lim_{N \rightarrow +\infty} [\ln x]_1^N = \lim_{N \rightarrow +\infty} \ln N = +\infty \quad (\text{i.e. limit does NOT exist})$$

$$\textcircled{2} \lim_{N \rightarrow +\infty} \int_1^N \frac{1}{x^2} dx = \lim_{N \rightarrow +\infty} \left[-\frac{1}{x}\right]_1^N = \lim_{N \rightarrow +\infty} 1 - \frac{1}{N} = 1$$

$$\textcircled{3} \lim_{N \rightarrow +\infty} \int_1^N \frac{1}{e^x} dx = \lim_{N \rightarrow +\infty} \left[-\frac{1}{e^x}\right]_1^N = \lim_{N \rightarrow +\infty} -\frac{1}{e^N} + \frac{1}{e}$$

Observation: $\lim_{x \rightarrow +\infty} f(x) = 0$ does NOT guarantee $\lim_{N \rightarrow +\infty} \int_a^N f(x) dx$ exists.

e.g. Find $\int_0^{+\infty} \frac{1}{(x+1)(3x+2)} dx$

Note: $(x+1)(3x+2)$ is a polynomial of degree 2.

$\frac{1}{(x+1)(3x+2)}$ decays as "fast" as $\frac{1}{x^2}$.

$$\lim_{N \rightarrow +\infty} \int_0^N \frac{1}{(x+1)(3x+2)} dx = \lim_{N \rightarrow +\infty} \int_0^N \frac{-1}{x+1} + \frac{3}{3x+2} dx$$

$$= \lim_{N \rightarrow +\infty} [-\ln|x+1| + \ln|3x+2|]_0^N$$

$$= \lim_{N \rightarrow +\infty} \ln \left| \frac{3N+2}{N+1} \right| - \ln 2$$

$$= \ln 3 - \ln 2$$

e.g. Find $\int_0^{+\infty} x e^{-2x} dx$

$$\lim_{N \rightarrow +\infty} \int_0^N x e^{-2x} dx$$

$$= \lim_{N \rightarrow +\infty} \int_0^N x d\left(-\frac{1}{2} e^{-2x}\right)$$

$$= \lim_{N \rightarrow +\infty} \left[-\frac{1}{2} x e^{-2x}\right]_0^N - \int_0^N -\frac{1}{2} e^{-2x} dx$$

$$= \lim_{N \rightarrow +\infty} \left[-\frac{1}{2} x e^{-2x}\right]_0^N + \left[-\frac{1}{4} e^{-2x}\right]_0^N$$

$$= \lim_{N \rightarrow +\infty} -\frac{1}{2} N e^{-2N} - \frac{1}{4} e^{-2N} + \frac{1}{4}$$

\swarrow go to 0 when $N \rightarrow +\infty$
 \searrow

$$= \frac{1}{4}$$

Exponential Growth:

• Simple Interest

Question: How much interest I can one get by putting money into a bank?

Answer: It depends on:

- how much one put (Principal P)
- how long one put (Time t (years))
- which bank one put (Interest rate r)

$$\therefore I = P \times t \times r$$

$$\text{Amount } A = P + I$$

$$= P \times (1 + t \times r)$$

• Compound Interest

Nothing, but calculating simple interest repeatedly!

e.g. $P = 10000$, $r = 5\%$, compound monthly (12 times per year),
deposit for 2 years.

$$A = 10000 \times \left(1 + \frac{5\%}{12}\right)^{12 \times 2}$$

e.g. P , r , compound monthly k times per year,
deposit for T years.

$$A = P \left(1 + \frac{r}{k}\right)^{kT}$$

Compound continuously \Rightarrow let $k \rightarrow +\infty$

$$\text{Amount} = \lim_{k \rightarrow +\infty} P \left(1 + \frac{r}{k}\right)^{kT}$$

$$= \lim_{k \rightarrow +\infty} P \times \left[\left(1 + \frac{1}{\frac{k}{r}}\right)^{\frac{k}{r}}\right]^{rT}$$

$$= P e^{rT}$$

(which is called exponential growth)

e.g. Suppose \$1000 is invested at an annual interest rate of 7% .
Compute the amount after 10 years if the interest is compounded

a) Annually ($k=1$) Ans: $A = 1000 \times (1 + \frac{7\%}{1})^{1 \times 10} \approx 1967.15$

b) Quarterly ($k=4$) Ans: $A = 1000 \times (1 + \frac{7\%}{4})^{4 \times 10} \approx 2001.60$

c) Monthly ($k=12$) Ans: $A = 1000 \times (1 + \frac{7\%}{12})^{12 \times 10} \approx 2009.66$

d) Continuously ($k \rightarrow \infty$) Ans: $A = 1000 \times e^{7\% \times 10} \approx 2013.75$

Exponential Growth / Decay :

In general ,

Q_0 = value at the beginning ($t=0$)

r = growth ($r > 0$) / decay ($r < 0$) rate

t = time (years)

$Q(t)$ = value at time t

$$= Q_0 e^{rt}$$

e.g. Depreciation (Depreciate continuously)

Value of a machine now = 20000

Rate of depreciation = 40% / year

$$\begin{aligned} \text{Value of a machine after 5 years} &= 20000 e^{-0.4 \times 5} \\ &\approx 2706.71 \end{aligned}$$

Application :

Quantities modelled in terms of exponential growth / decay :

- population growth of bacteria
- decay of radioactive substances
- concentration of drug in a patient's bloodstream.
- and etc

Relative Rates of Change

Think:

$Q(t)$: salary after working for t .

$Q'(t)$: rate of change of salary.

For example, $Q'(t) > 0$, that's good!

But, how good?

You have to compare it with the salary at that moment!

i.e. $\frac{Q'(t)}{Q(t)}$ and it is called its relative rate of change of Q .

$$\begin{aligned}\text{In general, relative rate of change of } Q(x) \text{ (w.r.t. } x) &= \frac{Q'(x)}{Q(x)} \\ &= \frac{d}{dx} \ln Q(x)\end{aligned}$$

Revisit of Exponential Growth / Decay

Suppose $Q(x)$ follows exponential growth / decay, i.e. $Q(x) = Q_0 e^{rx}$

Then relative rate of change of $Q(x)$

$$\begin{aligned}&= \frac{d}{dx} \ln Q(x) \\ &= \frac{d}{dx} \ln Q_0 e^{rx} \\ &= \frac{d}{dx} \ln Q_0 + rx \\ &= r\end{aligned}$$

\therefore If $Q(x)$ follows exponential growth / decay,
then relative rate of change of $Q(x) = r = \text{constant}$

Note, $Q'(x) = rQ(x)$,

e.g. suppose $r = -2\%$, $Q'(x) = -0.02Q(x)$,

it means the rate of change of $Q(x)$ is -2% .

Additional Applications of Definite Integration

e.g.

A new clinic has just established. Past statistics shows that fraction of patients who will still be receiving treatment at the clinic t months after their initial visit is given by the function $f(t) = e^{-t/10}$. The clinic initially accepts 500 patients plans to accept new patients at the constant rate $g(t) = 200$. Approximately, how many will be receiving treatment at the clinic 6 months from now?

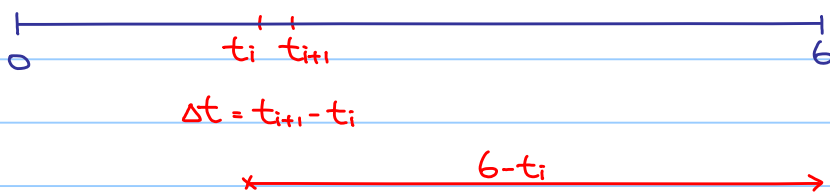
- $f(t)$ should be realized at the probability of a patient is still at the clinic after t months
(Exponential decay)

(I) First 500 patients :

Q: How many of them will remain at the clinic 6 months from now?

A: Simply $500 \times f(6)$
 ↑ ↑
 Number Probability

(II) Patients come later on :



Number of patients receiving from t_i to $t_{i+1} = 200(t_{i+1} - t_i) = 200\Delta t$

Number of them will remain at the clinic 6 months from now?

Summing up over $0 \leq t \leq 6$:

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n 200 f(6 - t_i) \Delta t = \int_0^6 200 f(6 - t) dt$$

∴ Number of patients after 6 months

$$= \underbrace{500f(6)}_{\text{(I)}} + \underbrace{\int_0^6 200f(6-t)dt}_{\text{(II)}}$$

$$= 500e^{-6/10} + \int_0^6 200e^{-(6-t)/10} dt$$

$$= 500e^{-6/10} + 200e^{-6/10} \int_0^6 e^{t/10} dt$$

$$= 500e^{-6/10} + 200e^{-6/10} \left[\frac{1}{10} e^{t/10} \right]_0^6$$

$$= 500e^{-6/10} + 200e^{-6/10} \left[\frac{1}{10} (e^{6/10} - 1) \right]$$

$$\approx 283$$

Generalization :

$P(t)$: value of a quantity

$P_0 = P(0)$: initial value

$R(t)$: renewal rate

$S(t)$: survival function

Then $P(T) = P_0 S(T) + \int_0^T R(t) S(T-t) dt$

e.g. It is estimated that t years from now, a nuclear plant will be producing nuclear waste at the rate 400 pounds/year. The waste decays exponentially at the rate of 2% per year. Assume that there is no radioactive waste at the beginning. How much radioactive waste will eventually accumulate?

$P(t)$: Amount of radioactive waste at time t (years)

$P_0 = 0$: Initial amount of radioactive waste

$R(t)$: 400

$S(t)$: $e^{-0.02t}$

Radioactive waste will eventually accumulate

$$= \lim_{T \rightarrow +\infty} \int_0^T 400 e^{-0.02(T-t)} dt$$

Ex :

$$= \lim_{T \rightarrow +\infty} 20000(1 - e^{-0.02T})$$

= 20000 pounds.