

Think: If $p(x)$ and $q(x)$ are polynomials

$$p(x) = a_m x^m + a_{m-1} x^{m-1} + \dots + a_1 x + a_0 \quad \text{with } a_m > 0 \quad (\text{i.e. } \deg p(x) = m)$$

$$q(x) = b_n x^n + a_{n-1} x^{n-1} + \dots + b_1 x + b_0 \quad \text{with } b_n > 0 \quad (\text{i.e. } \deg q(x) = n)$$

then find $\lim_{x \rightarrow +\infty} \frac{p(x)}{q(x)}$ for the following cases:

1) $\deg p(x) > \deg q(x)$ i.e. $m > n$

2) $\deg p(x) = \deg q(x)$ i.e. $m = n$

3) $\deg p(x) < \deg q(x)$ i.e. $m < n$

Ans:

$$\lim_{x \rightarrow +\infty} \frac{p(x)}{q(x)} = \begin{cases} +\infty & \text{if } \deg p(x) > \deg q(x) \\ \frac{a_m}{b_m} & \text{if } \deg p(x) = \deg q(x) \\ 0 & \text{if } \deg p(x) < \deg q(x) \end{cases}$$

Following this idea:

e.g. Find $\lim_{x \rightarrow +\infty} \frac{3x-1}{\sqrt{4x^2+2}}$

$$\lim_{x \rightarrow +\infty} \frac{3x-1}{\sqrt{4x^2+2}} \quad \leftarrow \text{roughly } \deg = 1$$

$$= \lim_{x \rightarrow +\infty} \frac{3 - \frac{1}{x}}{\sqrt{4 - \frac{2}{x}}}$$

$$= \frac{3}{2}$$

Constant e :

Consider a number $(1 + \frac{1}{m})^n$ that depends on m and n and then

1) fix m , say $m=100$, n is getting larger and larger.

$n=10$	$n=100$	$n=1000$	$\rightarrow +\infty$
$(1 + \frac{1}{m})^n = 1.01^{10}$	$(1 + \frac{1}{m})^n = 1.01^{100}$	$(1 + \frac{1}{m})^n = 1.01^{1000}$	$\rightarrow +\infty$

2) fix n , say $n=100$, m is getting larger and larger.

$m=10$	$m=100$	$m=1000$	$\rightarrow +\infty$
$(1 + \frac{1}{m})^n = 1.1^{100}$	$(1 + \frac{1}{m})^n = 1.01^{100}$	$(1 + \frac{1}{m})^n = 1.001^{100}$	$\rightarrow 1$

How about setting $m=n$ and let them become larger and larger?

$(1 + \frac{1}{n})^n \rightarrow ?$ as $n \rightarrow +\infty$ (i.e. limit exists?
something between $+\infty$ and 1 ??)

$n=10$	$n=100$	$n=1000$	$\rightarrow +\infty$
$(1 + \frac{1}{n})^n = 1.1^{10}$	$(1 + \frac{1}{n})^n = 1.01^{100}$	$(1 + \frac{1}{n})^n = 1.001^{1000}$	
≈ 2.59374	≈ 2.70481	≈ 2.71692	$\rightarrow 2.71828 \dots$

FACT (without proof)

$$\lim_{x \rightarrow +\infty} (1 + \frac{1}{x})^x \text{ exists!}$$

We define $e = \lim_{x \rightarrow +\infty} (1 + \frac{1}{x})^x \approx 2.71828$ (i.e. call the limit e)

Limits Involving e :

e.g. Find $\lim_{x \rightarrow +\infty} (1 + \frac{2}{x})^x$

$$\begin{aligned} \lim_{x \rightarrow +\infty} (1 + \frac{2}{x})^x &= \lim_{x \rightarrow +\infty} \left[(1 + \frac{1}{\frac{x}{2}})^{\frac{x}{2}} \right]^2 \\ &= e^2 \end{aligned}$$

$$\begin{aligned} \text{(let } y = \frac{x}{2}, \text{ as } x \rightarrow +\infty, y \rightarrow +\infty \\ \lim_{x \rightarrow +\infty} (1 + \frac{1}{\frac{x}{2}})^{\frac{x}{2}} = \lim_{y \rightarrow +\infty} (1 + \frac{1}{y})^y = e) \end{aligned}$$

e.g. Find $\lim_{x \rightarrow +\infty} \left(1 + \frac{1}{2x-1}\right)^x$

$$\begin{aligned}\lim_{x \rightarrow +\infty} \left(1 + \frac{1}{2x-1}\right)^x &= \lim_{x \rightarrow +\infty} \left(1 + \frac{1}{2x-1}\right)^{\frac{1}{2}(2x-1) + \frac{1}{2}} \\ &= \lim_{x \rightarrow +\infty} \left[\left(1 + \frac{1}{2x-1}\right)^{2x-1}\right]^{\frac{1}{2}} \cdot \left(1 + \frac{1}{2x-1}\right)^{\frac{1}{2}} \\ &= e^{\frac{1}{2}} \cdot 1 \\ &= e^{\frac{1}{2}}\end{aligned}$$

e.g. Find $\lim_{x \rightarrow -\infty} \left(1 + \frac{1}{x}\right)^x$

Let $y = -x$, as $x \rightarrow -\infty$, $y \rightarrow +\infty$

$$\begin{aligned}\lim_{x \rightarrow -\infty} \left(1 + \frac{1}{x}\right)^x &= \lim_{y \rightarrow +\infty} \left(1 - \frac{1}{y}\right)^{-y} \\ &= \lim_{y \rightarrow +\infty} \left(\frac{y}{y-1}\right)^y \\ &= \lim_{y \rightarrow +\infty} \left(1 + \frac{1}{y-1}\right)^{y-1} \cdot \left(1 + \frac{1}{y-1}\right) \\ &= e \cdot 1 \\ &= e\end{aligned}$$

Remark: From the above example, we know $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x$.

e.g. Find $\lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}}$.

Let $y = \frac{1}{x}$, as $x \rightarrow 0$, $y \rightarrow \infty$ (Not only $+\infty$, but also $-\infty$)

$$\lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = \lim_{y \rightarrow \infty} \left(1 + \frac{1}{y}\right)^y = e$$

Properties of e :

What so special with e ?

One is that : Consider a function $f(x) = e^x$

We have the expansion (Taylor Expansion) :

$$f(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$n! = 1 \times 2 \times 3 \times \dots \times n$$

and we know $f'(x) = \frac{df}{dx} = f(x)$ (i.e. the derivative of e^x is itself)

(Discuss later !)

$$\text{From } e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Roughly speaking : As $x \rightarrow +\infty$, e^x grows "faster" than any x^k , where $k > 0$

FACT (Without proof)

$$1) \lim_{x \rightarrow +\infty} x^k e^{-x} = \lim_{x \rightarrow +\infty} \frac{x^k}{e^x} = 0, \text{ for any } k > 0.$$

$$2) \lim_{x \rightarrow +\infty} p(x) e^{-x} = \lim_{x \rightarrow +\infty} \frac{p(x)}{e^x} = 0, \text{ for any polynomial } p(x).$$

Continuity :

A function $f(x)$ is said to be continuous at $x=c$ if $\lim_{x \rightarrow c} f(x) = f(c)$.



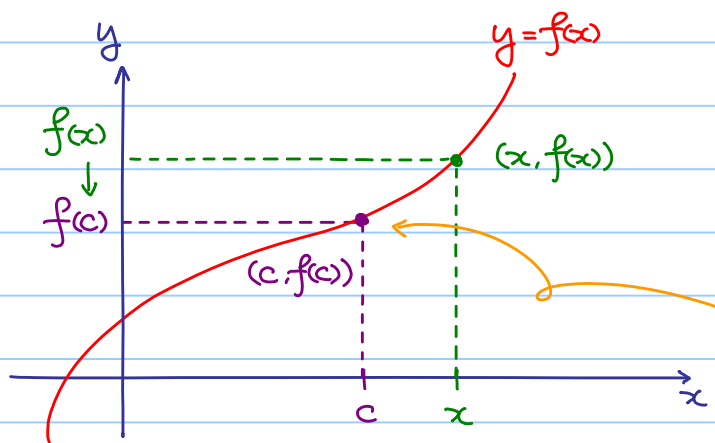
Idea :

③ they equal

$$\lim_{x \rightarrow c} f(x) = f(c)$$

① This limit exists

② f is well-defined at $x=c$



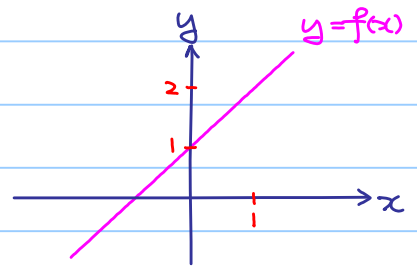
the curve does NOT
break up at the point $x=c$!

If a function is continuous at every point,
then f is called a continuous function.

e.g. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x+1$

① $\lim_{x \rightarrow 1} f(x) = 2$

② $f(1) = 2$

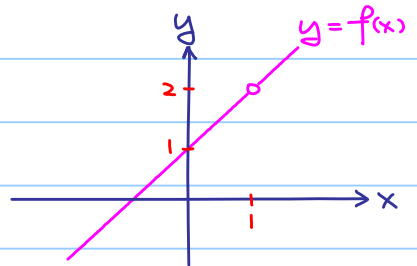


$\therefore f$ is discontinuous at $x=1$.

e.g. Let $f: \mathbb{R} \setminus \{1\} \rightarrow \mathbb{R}$ defined by $f(x) = \frac{x^2-1}{x-1}$, $x \neq 1$.

① $\lim_{x \rightarrow 1} f(x) = 2$

② $f(1)$ is NOT well-defined.



$\therefore f$ is discontinuous at $x=1$.

Recall :

$$\lim_{x \rightarrow c} f(x) = L \quad \text{if and only if} \quad \lim_{x \rightarrow c^+} f(x) = \lim_{x \rightarrow c^-} f(x) = L$$

Rewrite :

A function $f(x)$ is said to be continuous at $x=c$ if

$$\lim_{x \rightarrow c^+} f(x) = \lim_{x \rightarrow c^-} f(x) = f(c)$$

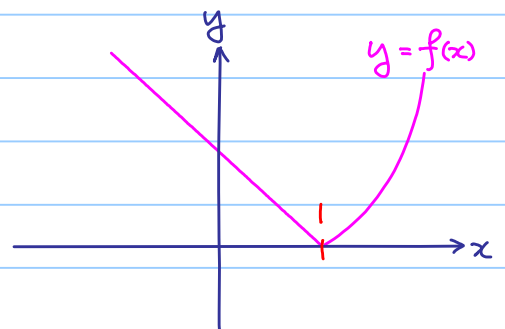
e.g. If $f(x) = \begin{cases} x^2-1 & \text{if } x \geq 1 \\ 1-x & \text{if } x < 1 \end{cases}$

① $\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} x^2-1 = 0$

② $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} 1-x = 0$

③ $f(1) = 1^2-1 = 0$

$\therefore f$ is continuous at $x=1$.



Absolute value :

$$|x| \stackrel{\text{def}}{=} \sqrt{x^2}$$

e.g. $|3| = \sqrt{3^2} = \sqrt{9} = 3$

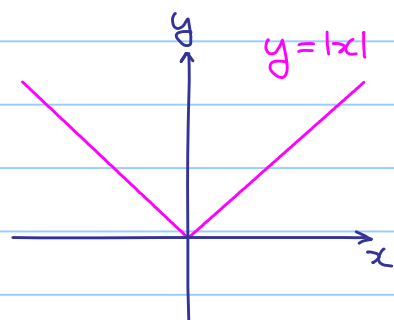
$$|-3| = \sqrt{(-3)^2} = \sqrt{9} = 3$$

$$|0| = 0$$

(Simply speaking : throw away the + or - sign.)

Rewrite :

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$



e.g. Prove $f(x) = |x|$ is continuous at $x=0$.

$$\textcircled{1} \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} x = 0$$

$$\textcircled{2} \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} -x = 0$$

$$\textcircled{3} f(0) = 0$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x) = f(0) = 0$$

$\therefore f(x)$ is continuous at $x=0$.

Further question : Is $f(x) = |x|$ a continuous function ?

Remarks:

1) We can further rewrite:

A function $f(x)$ is said to be continuous at $x=c$ if $\lim_{h \rightarrow 0} f(c+h) = f(c)$

(Hint: let $x=c+h$, as $h \rightarrow 0$, $x \rightarrow c$)

2) FACT (Without proof)

- polynomial function $p(x)$ is continuous everywhere.
- \sqrt{x} is continuous for $x \geq 0$
- If $f(x), g(x)$ are continuous, then $f(x) \pm g(x), f(x)g(x), \frac{f(x)}{g(x)}$ (when $g(x) \neq 0$) are continuous.
- If $f(x), g(x)$ are continuous, then $f(g(x))$ (when it is defined) is continuous.

e.g. $f(x) = \frac{2x^2+3}{x^2-3x+2}$ quotient of two polynomials (continuous functions)

$= \frac{2x^2+3}{(x-2)(x-1)}$ the denominator is nonzero when $x \neq 1$ or 2

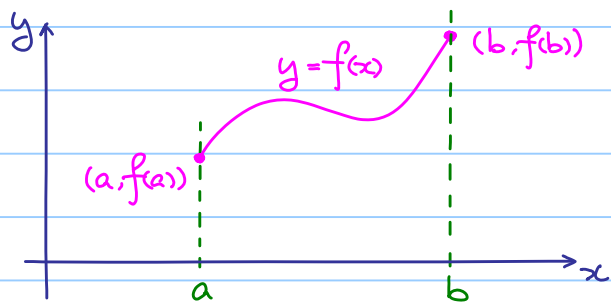
$\therefore f(x)$ is continuous everywhere except $x=1, 2$

Continuous on $[a, b]$:

Let $f: [a, b] \rightarrow \mathbb{R}$ be a function.

f is said to be continuous at $x=a$ if $\lim_{x \rightarrow a^+} f(x) = f(a)$

f is said to be continuous at $x=b$ if $\lim_{x \rightarrow b^-} f(x) = f(b)$



(We cannot talk about $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow b^+} f(x)$!)

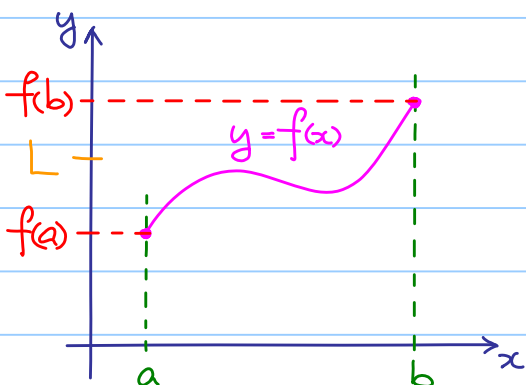
If $f: [a, b] \rightarrow \mathbb{R}$ is continuous at every point $x \in [a, b]$, then f is said to be continuous on $[a, b]$.

Mean Value Property :

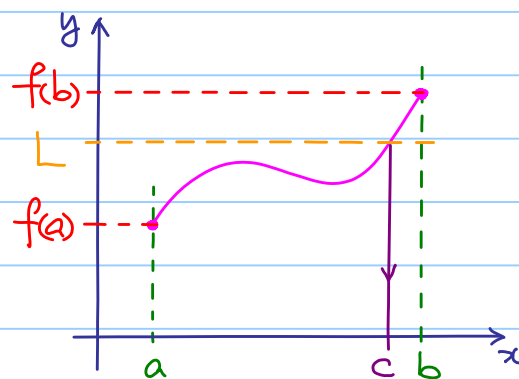
Suppose that f is continuous on $[a, b]$ and $f(a) < f(b)$.

Furthermore, if L is a real number such that $f(a) < L < f(b)$,

then there exists (at least one) $c \in (a, b)$ such that $f(c) = L$.



\Rightarrow



$$f(c) = L$$

Similar result holds for $f(a) > L > f(b)$. (What is the picture?)

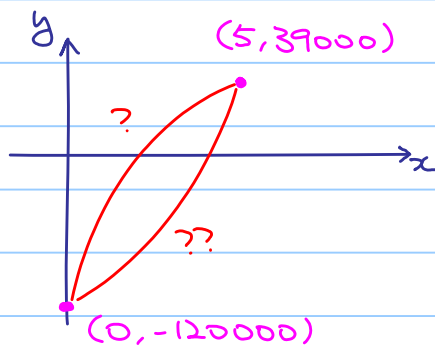
e.g. x : Number of products produced (in hundreds units)

$$\text{Revenue} = R(x) = 100x(400 - 3x^2)$$

$$\text{Cost} = C(x) = 120000 + 700x$$

$$\text{Profit} = P(x) = R(x) - C(x) = 100(-3x^3 + 393x - 1200)$$

- ① $P(0) = -120000 < 0$
- ② $P(5) = 39000 > 0$
- ③ $P(x)$ is a polynomial, so it is continuous everywhere,
In particular, it is continuous on $[0, 5]$



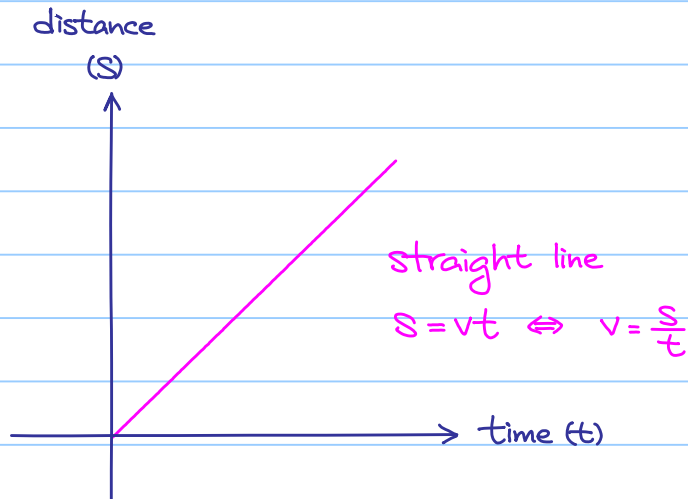
We do **NOT** know the shape of the graph, but we know it intersects the x -axis **at least once**.

i.e. $P(c) = 0$ (which means breakeven)
for some $c \in (0, 5)$

Conclusion: We do **NOT** know the shape of the graph, but we know it intersects the x -axis **at least once**, which may be enough for certain purpose.

Differentiation:

Recall: (average) speed = $\frac{\text{distance}}{\text{time}}$

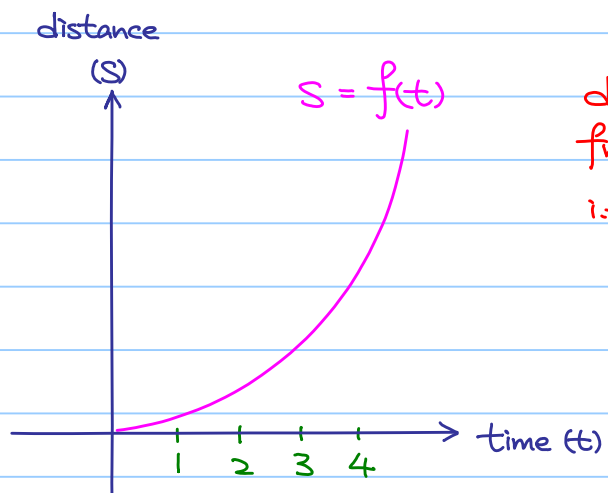


Note: constant speed!
speed = slope = v

Remark:

Using displacement and velocity if you know.

How about this case?



distance traveled from $t=0$ to $t=1$ < distance traveled from $t=3$ to $t=4$
i.e. speed is changing

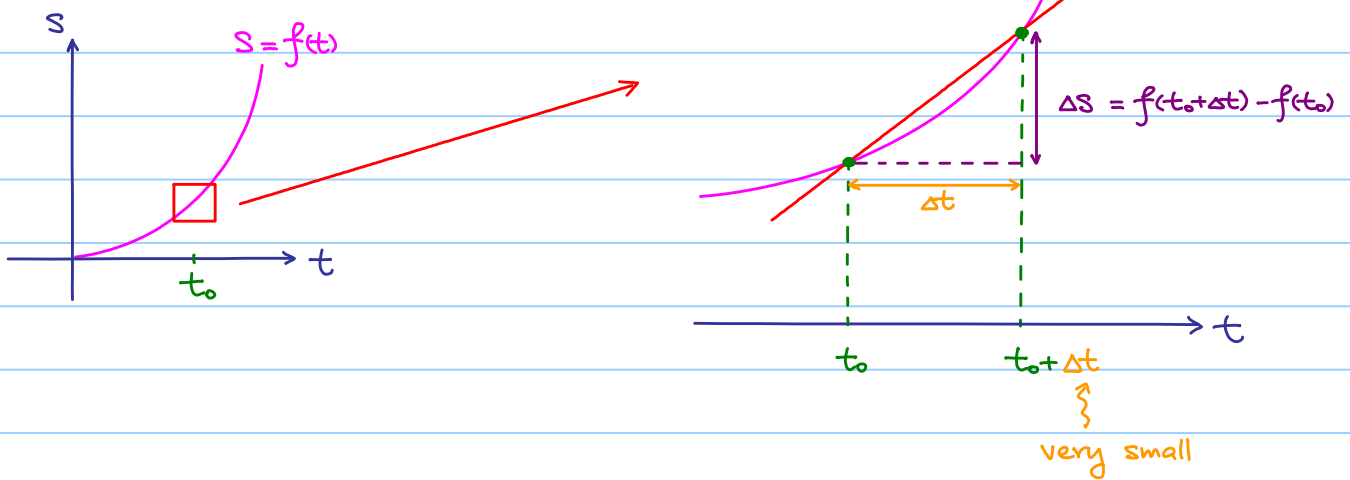
Speed is different at different moment.

Hold on!

What is the meaning of speed at a particular moment (instantaneous speed)?

We need a definition!

Instantaneous speed at $t=t_0$:



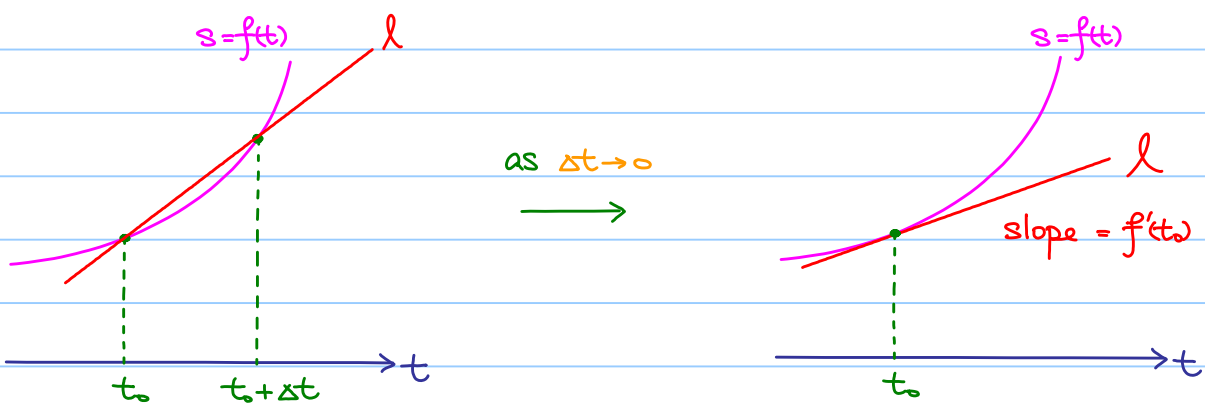
Average speed between t_0 and $t_0 + \Delta t$

$$= \frac{\text{change in distance}}{\text{change in time}} = \frac{\Delta s}{\Delta t} = \frac{f(t_0 + \Delta t) - f(t_0)}{\Delta t} = \text{slope of } l$$



Idea: Let Δt becomes smaller and smaller!

Instantaneous speed at $t=t_0$ is defined to be $\lim_{\Delta t \rightarrow 0} \frac{f(t_0 + \Delta t) - f(t_0)}{\Delta t}$
 (provided it exists, if so, it is denoted by $f'(t_0)$)



Note: When $\Delta t \rightarrow 0$, l becomes the tangent line at $t=t_0$, so
 slope of the tangent line at $t=t_0 = f'(t_0)$

e.g. If $s = f(t) = t^2$, find $f'(2)$ (instantaneous speed at $t=2$).

$$f'(2) = \lim_{\Delta t \rightarrow 0} \frac{f(2+\Delta t) - f(2)}{\Delta t}$$

$$= \lim_{\Delta t \rightarrow 0} \frac{(2+\Delta t)^2 - 2^2}{\Delta t}$$

$$= \lim_{\Delta t \rightarrow 0} \frac{4\Delta t + \Delta t^2}{\Delta t}$$

$$= \lim_{\Delta t \rightarrow 0} 4 + \Delta t = 4$$

In general, we have $y = f(x)$, fix x_0 .

Then $f'(x_0)$ means rate of change of y with respect to x at $x = x_0$.

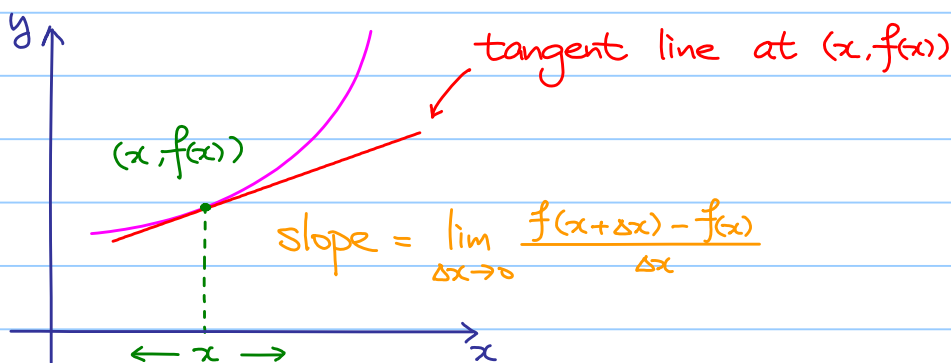
$f(x)$ is said to be differentiable at $x = x_0$ if

$$\lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \text{ exists (denoted by } f'(x_0) \text{)}$$

It is called the derivative of $f(x)$ at $x = x_0$.

Note: By definition, if $f(x_0)$ is NOT well-defined, we cannot define $f'(x_0)$, so $f(x)$ must NOT be differentiable at $x = x_0$.

Perform the previous step to different points :



Recall: What is a function ?

Roughly speaking, given an input x , return a value.

Now, we construct a new function, $f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$ (if exists)

(i.e. given an input x , return the slope of the tangent line at $(x, f(x))$)

e.g. If $f(x) = x^2$, find $f'(x)$

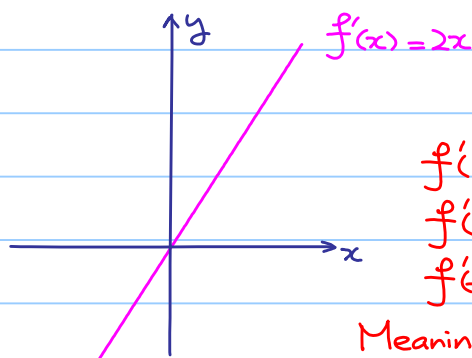
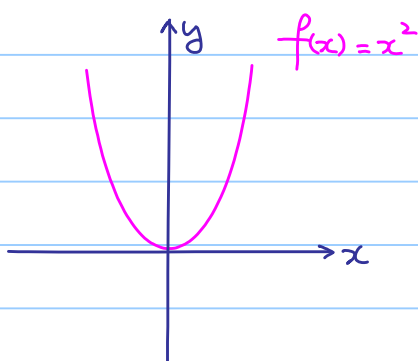
$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{2x\Delta x + \Delta x^2}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} 2x + \Delta x = 2x$$

Relation between the graphs of $f(x) = x^2$ and $f'(x) = 2x$:



$$f'(1) = 2$$

$$f'(0) = 0$$

$$f'(-1) = -2$$

Meaning ??

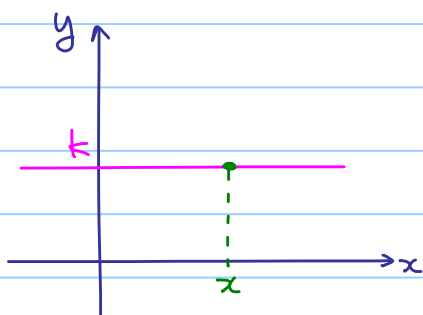
Notations :

$$y = f(x) = x^2$$

$$\frac{df}{dx} = \frac{dy}{dx} = f'(x) = 2x$$

$$\left. \frac{df}{dx} \right|_{x=3} = \left. \frac{dy}{dx} \right|_{x=3} = f'(3) = 2(3) = 6$$

e.g. If $f(x) = k$, where k is a constant, $f'(x) = ?$



Note : Slope of the tangent line at $(x, f(x)) = (x, k)$ is zero.
 $\therefore f'(x) = 0$

Concrete computation :

$$\begin{aligned} \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} &= \lim_{\Delta x \rightarrow 0} \frac{k - k}{\Delta x} \quad (\Delta x \neq 0) \\ &= \lim_{\Delta x \rightarrow 0} \frac{0}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} 0 = 0 \end{aligned}$$

Ex: Find $f'(x)$ if

(a) $f(x) = x$

Ans: $f'(x) = 1$

(b) $f(x) = x^3$

$f'(x) = 3x^2$

FACT (without proof)

If $f(x) = x^r$, where r is a real number,

then $f'(x) = rx^{r-1}$ whenever it is defined.

(Think : If $r = \frac{1}{2}$, $f(x) = \sqrt{x}$ which is defined when $x \geq 0$)

FACT: If $f(x)$ is differentiable at $x=x_0$,
then $f(x)$ is continuous at $x=x_0$.

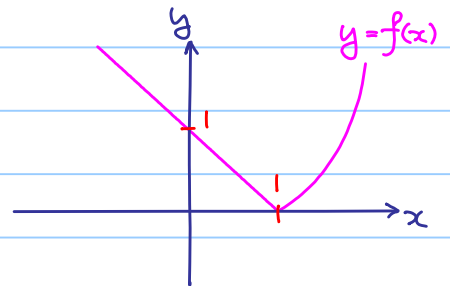
proof: By assumption, $\lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$ exists.
Also, we know $\lim_{\Delta x \rightarrow 0} \Delta x = 0$

$$\begin{aligned} \lim_{\Delta x \rightarrow 0} f(x_0 + \Delta x) - f(x_0) &= \lim_{\Delta x \rightarrow 0} \left(\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \cdot \Delta x \right) \\ &= \left(\lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \right) \cdot \left(\lim_{\Delta x \rightarrow 0} \Delta x \right) \quad \text{both exist} \\ &= f'(x_0) \cdot 0 = 0 \end{aligned}$$

$\therefore \lim_{\Delta x \rightarrow 0} f(x_0 + \Delta x) = f(x_0)$, so $f(x)$ is continuous at $x=x_0$.

However, the converse is **NOT** true.

e.g. If $f(x) = \begin{cases} x^2 - 1 & \text{if } x \geq 1 \\ 1 - x & \text{if } x < 1 \end{cases}$



$$\lim_{\Delta x \rightarrow 0^+} \frac{f(1 + \Delta x) - f(1)}{\Delta x} = \lim_{\Delta x \rightarrow 0^+} \frac{[(1 + \Delta x)^2 - 1] - [1^2 - 1]}{\Delta x} = \lim_{\Delta x \rightarrow 0^+} \frac{2\Delta x + \Delta x^2}{\Delta x} = 2$$

(it means we are looking at small but positive Δx)

$$\lim_{\Delta x \rightarrow 0^-} \frac{f(1 + \Delta x) - f(1)}{\Delta x} = \lim_{\Delta x \rightarrow 0^-} \frac{[1 - (1 + \Delta x)] - [1^2 - 1]}{\Delta x} = \lim_{\Delta x \rightarrow 0^-} \frac{-\Delta x}{\Delta x} = -1$$

(it means we are looking at small but negative Δx)

$$\lim_{\Delta x \rightarrow 0^+} \frac{f(1 + \Delta x) - f(1)}{\Delta x} \neq \lim_{\Delta x \rightarrow 0^-} \frac{f(1 + \Delta x) - f(1)}{\Delta x} \Rightarrow \lim_{\Delta x \rightarrow 0} \frac{f(1 + \Delta x) - f(1)}{\Delta x} \text{ does NOT exist!}$$

$\therefore f(x)$ is **NOT** differentiable at $x=1$.

Ex:

a) Show that $f(x)$ is continuous at $x=1$, i.e. $\lim_{x \rightarrow 1} f(x) = f(1)$.

(Therefore, $f(x)$ is continuous at $x=1$, but NOT differentiable at $x=1$.)

b) Show that $f(x)$ is differentiable everywhere except $x=1$, and

$$f'(x) = \begin{cases} 2x & \text{if } x > 1 \\ \text{undefined} & \text{if } x = 1 \\ -1 & \text{if } x < 1 \end{cases}$$

Differentiable function: If $f(x)$ is differentiable everywhere, then $f(x)$ is said to be a differentiable function.