Solution to Mid-Term 1520A

1. (a) $f'(x) = 100x^{99}$

(b) $h'(x) = \dfrac{1}{x^2+\sqrt{e^x+1}}\left(2x + \dfrac{e^x}{2\sqrt{e^x+1}}\right) = \dfrac{e^x+4x\sqrt{e^x+1}}{2\sqrt{e^x+1}\,(x^2+\sqrt{e^x+1})}$

2. $f(x) = \sqrt{(x+1)(x-2)^2} = \sqrt{(x+1)}\,|x-2| = \begin{cases} \sqrt{x+1}\,(x-2) & x \geq 2 \\ \sqrt{x+1}\,(2-x) & -1 \leq x < 2 \end{cases}$.

Hence,

$$\lim_{x\to 2^+} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2^+} \frac{(x-2)\sqrt{x+1}}{x-2} = \lim_{x\to 2^+} \sqrt{x+1} = \sqrt{3},$$

$$\lim_{x\to 2^-} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2^-} \frac{(2-x)\sqrt{x+1}}{x-2} = \lim_{x\to 2^-} (-\sqrt{x+1}) = -\sqrt{3}.$$

Since $\lim\limits_{x\to 2^+} \dfrac{f(x)-f(2)}{x-2} \neq \lim\limits_{x\to 2^-} \dfrac{f(x)-f(2)}{x-2}$, $f'(2)$ doesn't exist.

(b) $\lim\limits_{x\to 5} \dfrac{x-5}{\sqrt{x+4}-3} = \lim\limits_{x\to 5} \dfrac{(x-5)(\sqrt{x+4}+3)}{(\sqrt{x+4}-3)(\sqrt{x+4}+3)} = \lim\limits_{x\to 5} (\sqrt{x+4}+3) = 6.$

3. After differentiation with repect to t, we get

$$x\frac{dx}{dt} + 4y\frac{dy}{dt} = 0,$$

namely $\dfrac{dy}{dt} = -\dfrac{x}{4y}\dfrac{dx}{dt}$, put $x=2$, $y=1$, $\dfrac{dx}{dt}=5$ into it, we get $\dfrac{dy}{dt} = -\dfrac{5}{2}$. Hence the y-coordinate is decreasing at $\frac{5}{2}$ unit per second at this moment.

4. Since $f(x) = (1-x)e^{-x}$, we have $f'(x)=0 \Rightarrow x=1$; $f'(x)>0 \Rightarrow 0 \leq x < 1$; $f'(x)<0 \Rightarrow 1 < x \leq 2$. Therefore $f_{max} = f(1) = e^{-1}$, moreover since $f(0)=0$, $f(2) = 2e^{-2}$, $f_{min} = f(0) = 0$.

5. (a) $\int (x^2+e^x)dx = \frac{x^3}{3} + e^x$.

(b) $\int x\ln(x+1)dx = \int \ln(x+1)\,d\left(\frac{x^2}{2}\right) = \frac{x^2}{2}\ln(x+1) - \int \frac{x^2}{2}\,d\ln(x+1)$

$= \frac{x^2}{2}\ln(x+1) - \frac{1}{2}\int \frac{x^2}{x+1}dx$

$= \frac{x^2}{2}\ln(x+1) - \frac{1}{2}\int \frac{x^2-1+1}{x+1}dx$

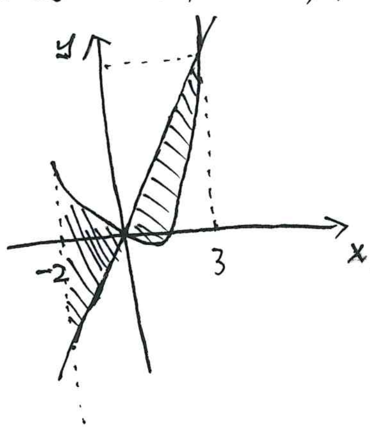
$= \frac{x^2}{2}\ln(x+1) - \frac{1}{2}\int (x-1)dx - \frac{1}{2}\int \frac{1}{x+1}dx$

$= \frac{x^2}{2}\ln(x+1) - \frac{1}{4}(x-1)^2 - \frac{1}{2}\ln(x+1) + C$

$= \frac{x^2-1}{2}\ln(x+1) - \frac{1}{4}(x-1)^2 + C$.

6. $\int_0^1 x(1+x^2)^{10}dx = \frac{1}{22}(1+x^2)^{11}\Big|_0^1 = \frac{2047}{22}$.

7. solving $\begin{cases} y=x^2-x \\ y=2x \end{cases}$, we get intesection points $(0,0)$, $(3,6)$.

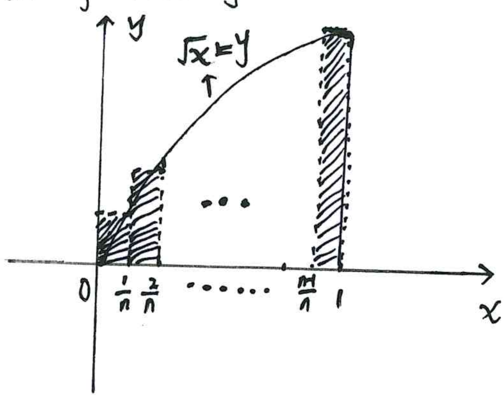


Hence the area

$A = \int_{-2}^{0}(x^2-x-2x)dx + \int_0^3 [2x-(x^2-x)]dx$

$= \frac{79}{6}$

8. $\frac{\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n}}{n\sqrt{n}} = \frac{1}{n}\sqrt{\frac{1}{n}} + \frac{1}{n}\sqrt{\frac{2}{n}} + \cdots + \frac{1}{n}\sqrt{\frac{n}{n}}$, this sum can be interpretated as the sum of area of rectangales shown below.



Hence $\lim\limits_{n\to\infty} \frac{\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n}}{n\sqrt{n}} = \int_0^1 \sqrt{x}\,dx = \frac{2}{3}x^{\frac{3}{2}}\Big|_0^1 = \frac{2}{3}$.

9. $\int e^{-x}\ln(1+e^x)dx = \int \ln(1+e^x)d(-e^{-x}) = -e^{-x}\ln(1+e^x) - \int(-e^{-x})d\ln(1+e^x)$

$$= -e^{-x}\ln(1+e^x) + \int \frac{1}{e^x+1}dx$$

$$= -e^{-x}\ln(1+e^x) + \int \frac{1+e^x-e^x}{e^x+1}dx$$

$$= -e^{-x}\ln(1+e^x) + x - \int \frac{e^x}{e^x+1}dx$$

$$= -e^{-x}\ln(1+e^x) + x - \ln(e^x+1) + C$$

$$= x-(1+e^{-x})\ln(1+e^x) + C.$$