THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1510A Calculus for Engineers (Fall 2014)
Comments on Homework 1
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Important Notice: For students from classes B and C, only the final score on the score sheet on top will count towards your grading. Please DO NOT refer to the score in the inside pages.

We will talk about some common crucial mistakes.

For Question 1,

1. Some students confuse **domain** with **range** of a function. Please read handouts of Chapter 1.

2. Some students confuse **open intervals** with **closed intervals**, and also **half-open intervals**. Please read handouts of Chapter 1.

For Question 2,

1. The same problems for Question 1 happen in Question 2.

2. For part (c), some students fail to realize $f(x) \geq 0$ if and only if

$$(3x - 4 \geq 0 \quad \text{and} \quad x^2 + x - 6 > 0) \quad \text{or} \quad (3x - 4 \leq 0 \quad \text{and} \quad x^2 + x - 6 < 0)$$

   Note that the denominator cannot be 0, for otherwise f is **undefined** at x.

3. Some students do not comprehend the meaning of **asymptotes**. Please read handouts of Chapter 4.

For Question 4,

1. Some students do not plot $f+g$ correctly. The value of $(f+g)(x)$ at x is $f(x)+g(x)$. Roughly speaking, it means that the 'heights' of $f(x)$ and $g(x)$ accumulate to the 'height' of $(f+g)(x)$.

2. Some students do not realize that $(-\infty, 0)$ is not in the domain of $f+g$. In fact, one can prove

$$D(f+g) = D(f) \cap D(g)$$

For Question 5,

1. Many students have the limit signs missed as they carry on calculations.

2. Some students test, say $n = 10$, in parts (a) and (b). In general, this is far less from enough to see the limits. Also, more than one value should be tested, for example, $n = 10^3, 10^4, 10^5$.

3. Some students attempt parts (a) and (b) analytically, but the reasonings are incorrect. For part (a), we are to show

$$\lim_{n\to\infty} \frac{3^n}{n!} = 0$$

   (a) Some students give reasons as follows:

   Proof. Since $\frac{3^n}{n!}$ is a decreasing sequence in n and $\frac{3^n}{n!} \geq 0$, we have

$$\lim_{n\to\infty} \frac{3^n}{n!} = 0$$

   . □

   This is insufficient to make the conclusion. Consider $1 + \frac{1}{n}$ is also a decreasing sequence in n and $1 + \frac{1}{n} \geq 0$, but $\lim_{n\to\infty} 1 + \frac{1}{n} = 1$. Instead, **Squeeze Theorem** should be used. Please read the solution.

   (b) Several students give reasons as follows:

   Proof. Since $\frac{3^n}{n!} = \frac{3}{n}\frac{3}{n-1}\cdots\frac{3}{2}\frac{3}{1}$, we have

$$\lim_{n\to\infty} \frac{3^n}{n!} = 0 \cdot 0 \cdot \ldots \cdot \frac{3}{2} \cdot \frac{3}{1} = 0$$

   □

   We have a theorem saying that if two limits exist, product of limits is equal to limit of product.

   Theorem 1. *If* $\lim_{n\to\infty} a_n = L_1$ *and* $\lim_{n\to\infty} b_n = L_2$*, then* $\lim_{n\to\infty} a_n b_n = L_1 L_2$

   One can show that the above is true if the number of terms is finite. But it is not applicable if the number of terms involved in the product also tends to infinity. Consider $e = \lim_{n\to\infty}(1 + \frac{1}{n})^n$, but $\lim_{n\to\infty} 1 + \frac{1}{n} = 1$.

For Question 7,

1. $\lim_{x\to a} f(x) = L$ does not imply $f(a) = L$. The implication is correct if and only if the function f is **continuous** at a. Please read handouts of Chapter 3.

2. In general, even if $\lim_{x\to a} f(x) = L$, f can be undefined at a. Note that as x approaches a, x possibly never attain a.

For Question 8,

1. $\frac{0}{0}$ is **indeterminate**, i.e. nothing can be concluded about the limit by just letting the numerator and the denominator approaching their limits separately. Please read handouts of Chapter 4. In general, it is not equal to 0, as a few students answer. Consider $f(x) = x$ and $g(x) = 2x$. Then $\lim_{x\to 0} f(x) = 0 = \lim_{x\to 0} g(x)$ but $\lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{1}{2}$. The indeterminate forms can sometimes be solved by eliminating removable terms from both the numerator and the denominator. In general, their values can be determined by the powerful **L'Hôpital's Rule**, which will be discussed later in the course.

2. Some students attempt part (a) with incorrect reasonings. For part (a), we are to show

$$\lim_{x\to 0} \frac{\sin(3\theta)}{\tan(\theta)} = 3$$

   (a) Some students give reasons as follows:

   Proof. As θ approaches 0, $\sin(\theta) = \theta = \tan(\theta)$, we have

$$\lim_{x\to 0} \frac{\sin(3\theta)}{\tan(\theta)} = \lim_{x\to 0} \frac{3\theta}{\theta} = 3$$

   . □

   Actually, one can in show $\sin(\theta) = \theta = \tan(\theta)$ is only true when $\theta = 0$. In fact, $\sin(\theta) < \theta < \tan(\theta)$ for $\theta \in (0, \frac{\pi}{2})$. As mentioned above, when θ approaches 0, θ possibly never attain 0, which $\sin(\theta) = \theta = \tan(\theta)$ is wrong.

   (b) Some students write the following:

   Proof. Since $|\sin(\theta)| \le |\theta| \le |\tan(\theta)|$ for $\theta \in (\frac{-\pi}{2}, \frac{\pi}{2})$, we have

$$\lim_{x\to 0} \frac{\sin(3\theta)}{\tan(\theta)} \le \lim_{x\to 0} \frac{3|\theta|}{|\theta|} = 3$$

   . □

   This is true, but it only shows a weaker result $\lim_{x\to 0} \frac{\sin(3\theta)}{\tan(\theta)} \le 3$.

For Question 10,

1. Some students do not comprehend the meaning of the limit of $f(x)$ **does not exist** and the limit of $f(x)$ is ∞ or $-\infty$ as x approaches ∞. Please read handouts of Chapter 2. To summarize,

   (a) $\lim_{x\to\infty} f(x) = \infty$ if and only if
   for any given $K > 0$, there exists $N > 0$ such that $f(x) > K$ for $x > N$.
   Roughly speaking, any $K > 0$ cannot bound the tail of $f(x)$ from above. Similar for the limit of $f(x)$ is $-\infty$ as x approaches ∞.

   (b) The limit of $f(x)$ does not exist as x approaches ∞ because the tail of $f(x)$ is irregular, either approaching a real value, ∞ or $-\infty$. Periodic functions in this question is a good example.

2. To prove a limit does not exist, you may find the following **Divergence Criterion** useful:

   Theorem 2. *Let c be either a real number, ∞ or $-\infty$. If there exists two sequences (x_n) and (y_n) such that $x_n \to c$ and $y_n \to c$, but $\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)$, then $\lim_{x\to c} f(x)$ does not exist.*

   Students are invited to try two such limits for each of parts (a) and (b).