§ Series

Given a sequence (a_n) of numbers, we can consider:

$$s_1 = a_1, \quad s_2 = a_1 + a_2, \quad s_3 = a_1 + a_2 + a_3 = \sum_{k=1}^{3} a_k,$$

$$s_4 = \sum_{k=1}^{4} a_k \ (= a_1 + a_2 + a_3 + a_4), \ \ldots,$$

$$s_n = \sum_{k=1}^{n} a_k \quad (n \in \mathbb{N}).$$

so we get a new sequence $(s_n)_{n \in \mathbb{N}}$.

This sequence (s_n) is called the <u>series</u> obtained from the sequence (a_n), and s_n is called the <u>n^{th} partial sum</u> of (a_n).

We also denote (s_n) by $\sum_{n=1}^{\infty} a_n$. $\sum_{k=1}^{\infty} a_k$

If $\lim_{n\to\infty} s_n = l \in \mathbb{R}$, ($\infty$, or $-\infty$), we say that <u>the series</u> (s_n), or $\sum_{n=1}^{\infty} a_n$, <u>converges to l</u>, (<u>diverges</u> to ∞ (or $-\infty$)) and write

$$\sum_{n=1}^{\infty} a_n = l.$$

Ex 1 (The geometric progression/series)

Let $a_n = \frac{1}{2^n}$, $n \in \mathbb{N}$.

Then $\quad s_n = \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n}$

$\therefore \quad \frac{1}{2} s_n = \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} + \frac{1}{2^{n+1}}$

$\therefore \quad (1-\frac{1}{2}) s_n = \frac{1}{2} - \frac{1}{2^{n+1}}$

i.e. $\quad s_n = \frac{1}{1/2}\left(\frac{1}{2} - \frac{1}{2^{n+1}}\right) = 1 - \frac{1}{2^n}$.

Hence,

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

Similarly, if $d \in (0,1)$, then

$$\sum_{n=1}^{\infty} d^n = \lim_{n\to\infty} \frac{d - d^{n+1}}{1-d} = \frac{d}{1-d}.$$

$\sum_{k=1}^{n} d^k = \frac{d-d^{n+1}}{1-d} \xrightarrow[n\to\infty]{} \infty$ if $d>1$

If $d \geq 1$, then $\sum_{n=1}^{\infty} d^n = \infty$.

Ex. 2 For each $n \in \mathbb{N}$,

$$1+2+\cdots+n = \frac{n(n+1)}{2}.$$

Hence

$$\sum_{n=1}^{k} \frac{1}{1+2+\cdots+n} = \sum_{n=1}^{k} \frac{1}{n(n+1)/2} = \sum_{n=1}^{k} \frac{2}{n(n+1)}$$

$$= \sum_{n=1}^{k} \left(\frac{2}{n} - \frac{2}{n+1} \right)$$

$$= \frac{2}{1} - \frac{2}{1+1} + \frac{2}{2} - \frac{2}{2+1} + \frac{2}{3} - \frac{2}{3+1} + \cdots + \frac{2}{k} - \frac{2}{k+1}$$

$$= 2 - \frac{2}{k+1},$$

i.e. $\displaystyle\sum_{n=1}^{k} \frac{1}{1+2+\cdots+n} = 2 - \frac{2}{k+1} \to 2$ as $k \to \infty$.

$$\therefore \quad \sum_{n=1}^{\infty} \frac{1}{1+2+\cdots+n} = 2.$$

(We can consider ~~series~~ sequences of complex numbers.)

Simple Facts

1. If $\sum_{n=1}^{\infty} a_n$ converges (to a finite/real number),

then $\lim_{n\to\infty} a_n = 0$. (Note that $s_n - s_{n-1} = a_n$.)

2. If $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ (both) converge, then

$\sum_{n=1}^{\infty} (a_n + b_n)$ converges to $\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$.

In this case, we write

$$\sum_{n=1}^{\infty}(a_n+b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n .$$

(Here, $\sum_{n=1}^{\infty} a_n$ also means the limit ℓ of the

partial sums $s_n = \sum_{k=1}^{n} a_k$. Similar remark applies also to

$\left(\sum_{k=1}^{n} b_k\right)_{n\in\mathbb{N}}$, etc.)

3. $\sum_{n=1}^{\infty} k\, a_n = k \sum_{n=1}^{\infty} a_n$ if $k\in\mathbb{R}$ and $\sum_{n=1}^{\infty} a_n$ converges.

We can consider sequence of functions, say on $\mathbb{R}$. For example, $(f_n)_{n\in\mathbb{N}}$ where

$$f_n(x) = \frac{x^n}{n!}, \quad x \in \mathbb{R}$$

i.e. the sequence

$$\frac{1}{0!}=1, \frac{x}{1!}, \frac{x^2}{2!}, \frac{x^3}{3!}, \dots, \frac{x^n}{n!}, \dots .$$

And we can consider the series obtained from a sequence of functions, e.g.

$$\sum_{n=1}^{\infty} f_n(x),$$

which means the sequence $(s_n(x))_{n\in\mathbb{N}}$ of partial sums, i.e.

$$s_n(x) = \sum_{k=1}^{n} f_k(x).$$

In case $f_n(x) = x^n/n!$, then

$$s_n(x) = x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}$$

Again, if the series converges at x, then we write $$\sum_{n=1}^{\infty} f_n(x)$$ to mean both the sequence $(s_n(x))_{n\in\mathbb{N}}$ and the limit $\lim_{n\to\infty} s_n(x)$. So

$$\sum_{n=1}^{\infty} f_n(x)$$

is a function, whose domain is the set of all x $(\in \mathbb{R})$ such that $\lim_{n\to\infty} s_n(x)$ exists as a real number.

Sometimes we start with $n=0$, i.e. we consider $f_0, f_1, f_2, \cdots$ and write $\sum_{n=0}^{\infty} f_n(x)$.

$\boxed{\text{Important Facts}}$

For all $x \in \mathbb{R}$, $e^x = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!},$$

$$\cos x = 1 + \sum_{k=1}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} . = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

$0! = 1$

$(-1)^0 = (\sin)'(0)$

$(-1)^1 = (\sin)^{(3)}(0) = (-\cos)(0)$

$(\sin)''(0) = (-\sin)(0) = 0$

$(-1)^0 \frac{x^1}{1!} + 0 \cdot \frac{x^2}{2!} + (-1)^1 \frac{x^3}{3!} + 0 \cdot \frac{x^4}{4!} + (-1)^2 \frac{x^5}{5!} + \cdots$

$x_0 = 0$

Very **often**, we have (so-called Taylor series expansion of f):

Taylor expansion Thm

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \cdots$$

in a certain **(maybe small)** neighbourhood of x_0 **(i.e. for x suff. close to x_0)**,
which is usually an interval centered at x_0 (called interval of convergence)

$x_0 = 0$, Maclaurin Series

1. Taylor Series & Maclaurin Series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n, \quad x \in I;$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n, \quad x \in J.$$

True under fairly gen. cond. on f

(1) why? what lead us to consider these?

(a) Insight from polynomials (formal generalization)

(b) Insight from approximation

(i) linear approx.; exact formula (with remainder)

(ii) quadratic approx.; exact formula (with remainder)

(iii) approx. by polynomials of degree n; exact formula (with remainder)

(2) (possible) operations with them:

Sum
product
Differentiation
Integration

§3 Taylor Series & Maclaurin Series

Theorem 1 Let $n \in \mathbb{N} \cup \{0\}$, and $a < b$ in $\mathbb{R}$ (or $a > b$ in $\mathbb{R}$). Suppose $f : [a,b] \to \mathbb{R}$, and its derivatives $f', f'', \cdots, f^{(n)}$ are continuous on $[a,b]$, and $f^{(n)}$ is differentiable on (a,b) [or replaced by $[b,a]$ or (b,a), respectively]. Then there exists $c \in (a,b)$ such that

$\circledast\ f^{(0)} \stackrel{\text{def}}{=} f.$

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

Corollary Let $n, a, b, f, f', \cdots, f^{(n)}$ be as in the preceding Thm 1. Then, for each $x \in (a,b)$ or (b,a), there exists $c \in (a,x)$ or (x,a) such that

$$f(x) = \underbrace{f(a) + \frac{f'(a)}{1!}(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n}_{n^{\text{th}} \text{ Taylor polynomial about } a \text{ of the function } f} + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}}_{\text{Lagrange's form of the remainder}}.$$

[Note that c may depend on x.]

Proof of Thm 1.

Illustrations

(1) $n = 0$ — mean value thm.

(2) $n = 1$. Let $a < b$ and define $F : [a,b] \to \mathbb{R}$ by

$$F(t) = f(t) - [f(a) + f'(a)(t-a) + K(t-a)^2], \quad t \in [a,b],$$

where K is a constant number such that $F(b) = 0$ $\{$i.e. $K = [f(b) - f(a) - f'(a)(b-a)]/(b-a)^2\}$.

Then $F(a) = 0 = F'(a)$, $F(b) = 0$. By applying the mean value theorem twice, we see that there exist $c_1 \in (a,b)$ such that $F'(c_1) = 0$, and $c_2 \in (a, c_1)$ such that $F''(c_2) = 0$. Hence $0 = f''(c_2) - (2!)K$, i.e. $K = \frac{1}{2!} f''(c_2)$. Thm 1 follows.

Thm 2. Let $n \in \mathbb{N}$, and $a < b$ in $\mathbb{R}$ (or $a > b$ in $\mathbb{R}$). Suppose $f : [a,b] \to \mathbb{R}$, and its derivatives $f', f'', \cdots, f^{(n-1)}$ are continuous on $[a,b]$ (or $[b,a]$) and $f^{(n-1)}$ is differentiable on (a,b) [or (b,a)]. Let $\alpha \in \mathbb{R}$ and define

$$\varphi_\alpha : [a,b] \to \mathbb{R} : x \mapsto f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1} + \alpha\frac{(x-a)^n}{n!}.$$

Then $\varphi_\alpha(a) = f(a)$, $\varphi_\alpha'(a) = f'(a)$, $\cdots$, $\varphi_\alpha^{(n-1)}(a) = f^{(n-1)}(a)$, $\varphi_\alpha^{(n)}(a) = \alpha$

and φ_α is a polynomial of degree no more than n. Moreover,

when x is sufficiently close to a ($x \in (a,b]$),

$$|f(x) - \varphi_\alpha(x)| > |f(x) - \varphi_{f^{(n)}(a)}(x)|, \quad \alpha \neq f^{(n)}(a).$$

Proof of Thm 2

Illustrations

(1) $n = 1$. Observe that

$$\lim_{x \to a^+}\left|\frac{f(x) - \varphi_\alpha(x)}{x-a}\right| = \left|\lim_{x \to a^+}\frac{f(x) - \varphi_\alpha(x)}{x-a}\right|$$

$$= \left|\lim_{x \to a^+}\frac{f(x) - [f(a) + \alpha(x-a)]}{x-a}\right| = \left|\lim_{x \to a^+}\left[\frac{f(x) - f(a)}{x-a} - \alpha\right]\right|$$

$$= |f'(a) - \alpha| = \begin{cases} > 0, & \text{if } \alpha \neq f'(a), \\ = 0, & \text{if } \alpha = f'(a). \end{cases}$$

Thus, when $x \in (a,b]$ is suff. close to a, and $\alpha \neq f'(a)$,

$$\frac{|f(x) - \varphi_\alpha(x)|}{|x-a|} = \left|\frac{f(x) - \varphi_\alpha(x)}{x-a}\right| > \frac{|f'(a) - \alpha|}{2} > \underbrace{\left|\frac{f(x) - \varphi_{f'(a)}(x)}{x-a}\right|}_{= \left|\frac{f(x)-f(a)}{x-a} - f'(a)\right|}$$

$$= \frac{|f(x) - \varphi_{f'(a)}(x)|}{|x-a|},$$

$\to \lim_{x \to a^+}\frac{f(x)-f(a)}{x-a} = f'(a)$

so $|f(x) - \varphi_\alpha(x)| > |f(x) - \varphi_{f'(a)}(x)|$.

(2) $n=2$. Observe that

$$\lim_{x\to a^+}\left|\frac{f(x)-\varphi_\alpha(x)}{(x-a)^2}\right| = \left|\lim_{x\to a^+}\frac{f(x)-\varphi_\alpha(x)}{(x-a)^2}\right|$$

$$= \left|\lim_{x\to a^+}\frac{f'(x)-\varphi_\alpha'(x)}{2(x-a)}\right| \qquad \text{(by L'Hospital rule)}$$

$$= \left|\lim_{x\to a^+}\left[\frac{f'(x)-f'(a)}{2\,(x-a)} + \frac{f'(a)-\varphi_\alpha'(x)}{2\,(x-a)}\right]\right|$$

$$= \left|\frac{f''(a)}{2!} - \frac{\alpha}{2!}\right| = \frac{1}{2}\left|f''(a)-\alpha\right| = \begin{cases} >0, & \text{if } \alpha\neq f''(a) \\ =0, & \text{if } \alpha = f''(a). \end{cases}$$

The rest is ditto case (1) $n=1$.

Thm 3 Let $n\in\mathbb{N}$, and $a<b$ in $\mathbb{R}$ (or $a>b$ in $\mathbb{R}$). Suppose $f:[a,b]\to\mathbb{R}$, and its derivatives $f', f'', \dots, f^{(n-1)}, f^{(n)}$ are continuous on $[a,b)$ {or $(b,a]$}. Let $\varphi:[a,b]\to\mathbb{R}$ be defined by:

$$\varphi(x) = f(a)+f'(a)(x-a)+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Then $\displaystyle\lim_{x\to a^+}\frac{f(x)-\varphi(x)}{(x-a)^n}=0.$

Pf By L'Hospital rule.

What operations can be carried on Taylor series? (Or, power series for that matter.) Many. For example, we know that

$$(*) \quad \sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}, \quad x \in \mathbb{R}.$$

Termwise diff. on the r.h.s. of (*) gives

$$\sum_{k=0}^{\infty} \frac{d}{dx}\left[(-1)^k \frac{x^{2k+1}}{(2k+1)!}\right] = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

which is exactly the Taylor series expansion of $\cos x$, the derivative of $\sin x$, the l.h.s of (*).
It suggests that we can perform termwise diff. on Taylor series expansions of functions.

Similarly, termwise integration on the r.h.s. of (*) gives

$$\sum_{k=0}^{\infty} \int_0^t (-1)^k \frac{x^{2k+1}}{(2k+1)!}\, dx$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k+2}}{(2k+2)!} = -\sum_{l=1}^{\infty} (-1)^l \frac{t^{2l}}{(2l)!} \quad (l = k+1)$$

$$= 1 - \sum_{l=0}^{\infty} (-1)^l \frac{t^{2l}}{(2l)!} = 1 - \cos t = -\cos x \Big|_{x=0}^{x=t}$$

$$= \int_0^t \sin x\, dx.$$

It suggests that we can perform terwise integration on Taylor series expansions of functions

Recall that

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}, \quad t \in \mathbb{R}$$

Hence $$e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}, \quad x \in \mathbb{R} \quad (**)$$

Termwise integration gives:

$$\int_1^t e^{x^2} dx = \sum_{n=0}^{\infty} \frac{1}{n!} \int_1^t x^{2n} dx$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{2n+1} x^{2n+1} \Big|_1^t ,$$

i.e. $$\int_1^t e^{x^2} dx = \sum_{n=0}^{\infty} \frac{-1}{n!} \frac{1}{2n+1} + \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{2n+1} t^{2n+1}$$

which gives a way to compute a numerical approximation of $\int_1^t e^{x^2} dx$.

One may ask: whether (**) is the Taylor series expansion of the function e^{x^2}? This can be confirmed by direct computation:

$$\frac{1}{k!} \frac{d^{(k)} e^{x^2}}{dx^k} \Big|_{x=0} = \begin{cases} 0, & \text{if } k = 2n-1, \ n = 1, 2, \cdots, \\ \frac{1}{n!} & \text{if } k = 2n, \ n = 0, 1, \cdots . \end{cases}$$

But, there is a (general) uniqueness theorem which ~~asserts~~ implies that (**) is indeed the Taylor series expansion of the function e^{x^2}.

Can we multiply two Taylor series? Check the case of $\sin x \cos x$ —— see whether you get $\frac{1}{2}\sin(2x)$ in some way. It suggests that indeed you can.

Indeed, under fairly general conditions, the above operations are all feasible.

(More) Example 1

$$\because \quad e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}, \quad x \in \mathbb{R}$$

$$\therefore \frac{1}{x}\left[e^{x^2} - 1\right] = x^{-1} \sum_{n=1}^{\infty} \frac{x^{2n}}{n!}, \quad x \in \mathbb{R}\setminus\{0\}$$

$$= \sum_{n=1}^{\infty} \frac{x^{2n-1}}{n!}$$

i.e. (4) $$\frac{e^{x^2}-1}{x} = x \sum_{n=1}^{\infty} \frac{x^{2(n-1)}}{n!} \quad \text{for all } x \in \mathbb{R}\setminus\{0\}.$$

By thm of Power Series $\lim_{x\to 0} \sum_{n=1}^{\infty} \frac{x^{2n-1}}{n!} = \sum_{n=1}^{\infty} \frac{0^{2n-1}}{n!} = 0$

Now, for each $x \in \mathbb{R}$,

$$0 \leq \sum_{n=1}^{k} \frac{x^{2(n-1)}}{n!} \leq \sum_{n=0}^{k-1} \frac{x^{2n}}{n!} \leq e^{x^2}.$$

Hence $0 \leq \lim_{k\to\infty} \sum_{n=1}^{k} \frac{x^{2(n-1)}}{n!} \leq e^{x^2}$ for all $x \in \mathbb{R}$. Therefore, by (4),

$$\lim_{x\to 0} \frac{e^{x^2}-1}{x} = 0.$$

This last result can be confirmed by L'Hospital rule.

Example 2

$$e^{x^2} = 1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{n!}, \qquad x \in \mathbb{R}$$

$$\frac{1}{x^2}\left(e^{x^2}-1\right) = x^{-2} \sum_{n=1}^{\infty} \frac{x^{2n}}{n!}, \qquad x \in \mathbb{R}\setminus\{0\}$$

$$= \sum_{n=1}^{\infty} x^{-2} \frac{x^{2n}}{n!}$$

$$= \sum_{n=1}^{\infty} \frac{x^{2(n-1)}}{n!} = 1 + \sum_{k=0}^{\infty} \frac{x^{2k}}{(k+1)!}$$

One can shorten this by making use a theorem on power series.

$$= 1 + x^2 \underbrace{\sum_{k=1}^{\infty} \frac{x^{2(k-1)}}{(k+1)!}}_{\text{converges to a finite number (for each } x \in \mathbb{R})}$$

$$\& \left| \sum_{k=1}^{\infty} \frac{x^{2(k-1)}}{(k+1)!} \right| \leq \sum_{l=0}^{\infty} \frac{x^{2l}}{(l+2)(l+1)[l!]} \leq \frac{1}{2} \sum_{l=0}^{\infty} \frac{x^{2l}}{l!} \leq \frac{1}{2} e^{x^2} \quad (4)$$

$$\therefore \lim_{x\to 0} \frac{e^{x^2}-1}{x^2} = 1 \qquad \left(\text{as } \lim_{x\to 0}\left[x^2 \sum_{k=1}^{\infty} \frac{x^{2(k-1)}}{(k+1)!}\right] = 0. \right)$$

by Sandwich theorem (*)

---

obtained by Taylor's thm

The result is confirmed by application of L'Hospital rule:

can be obtained by realising the lit $= \left.\frac{d e^y}{dy}\right|_{y=0} = 1$

$$\lim_{x\to 0} \frac{e^{x^2}-1}{x^2} = \lim_{x\to 0} \frac{2x e^{x^2}}{2x} = \lim_{x\to 0} e^{x^2} = e^0 = 1.$$

(*) $\because\ 0 \leq x^2 \sum_{k=1}^{\infty} \frac{x^{2(k-1)}}{(k+1)!} \leq \frac{1}{2} x^2 e^{x^2}$ by (4),

and $\lim_{x\to 0} \frac{1}{2} x^2 e^{x^2} = \frac{1}{2}\left(\lim_{x\to 0} x^2\right)\left(\lim_{x\to 0} e^{x^2}\right) = 0,$

$\therefore$ (by Sandwich thm) $\lim_{x\to 0}\left[x^2 \sum_{k=1}^{\infty} \frac{x^{2(k-1)}}{(k+1)!}\right] = 0.$

Thm For a (power) series of the form $\sum_{k=0}^{\infty} a_k (x-a)^k$, where a, a_k, $k \in \mathbb{N} \cup \{0\}$, are real numbers, exactly one of the following is true:

(i) the series converges only for $x = a$.

(ii) The series converges for all $x \in \mathbb{R}$.

(iii) There exists uniquely a positive real number R such that the series converges at every point in $(a-R, a+R)$, and diverges at every point in $(-\infty, a-R) \cup (a+R, \infty)$. This unique number R is called the radius of convergence of the series.

By convention we define $R=0$ in case (i), $R=\infty$ in case (ii) and call them radius of convergence of the series too.

Thm (i) Suppose $\lim_{n\to\infty} |a_n|^{1/n} = L$, or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$ ⊛, $L \in (0,\infty)$. Then the radius R of convergence of the series $\sum_{k=0}^{\infty} a_k (x-a_0)^k$ is given by: $R = 1/L$.

(ii) If the above limit $L = 0$, then the radius R of convergence of the above series is given by: $R = \infty$; i.e. the series converges everywhere in $\mathbb{R}$.

(iii) If the above limit $L = \infty$, then the radius R of convergence of the above series is given by: $R = 0$; i.e. the series converges only for $x = a$.

⊛ Note When $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$, then $\lim_{n\to\infty} |a_n|^{1/n} = L$ too (were $L \in \mathbb{R}$, or $L = \infty$). But not the converse (in general).

Term-by-term
<u>Thm</u> (Termwise differentiation)

If $\sum_{k=0}^{\infty} a_k x^k$ has radius of convergence $R>0$, then $f(x)=\sum_{k=0}^{\infty} a_k x^k$ is diff. on $(-R,R)$ and

$$f'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}, \quad x \in (-R,R).$$

<u>Note</u> Above we make the convention $\infty > 0$.

(Uniqueness)
<u>Thm</u> Suppose that $f(x)=\sum_{k=0}^{\infty} a_k x^k$ and $g(x)=\sum_{k=0}^{\infty} b_k x^k$ converge for all $x \in (-R,R)$. Suppose that there are infinitely many x_n, $n \in \mathbb{N}$, and a real positive number K ($<R$) such that $|x_n| \leq K$ for all $n \in \mathbb{N}$, and $f(x_n) = g(x_n)$ for all $n \in \mathbb{N}$.
Then $a_n = b_n$ for all $n \in \mathbb{N} \cup \{0\}$, and $f(x)=g(x)$ for all $x \in (-R,R)$.

Suppose ~~Supp~~ $f(x) = g(x)$ for all $x \in \mathbb{R}$.

Term-by-term
<u>Thm</u> (Termwise integration)

If $\sum_{k=0}^{\infty} a_k x^k$ has radius of convergence $R>0$, then $f(x)=\sum_{k=0}^{\infty} a_k x^k$ is continuous on any $[a,b] \subset (-R,R)$, and

$$\int_a^b f(x)\,dx = \sum_{k=0}^{\infty} a_k \left(\int_a^b x^k\,dx \right) = \sum_{k=0}^{\infty} \frac{a_k}{k+1} \left(b^{k+1} - a^{k+1} \right).$$

S. Ponnusamy

<u>Foundations of Mathematical Analysis</u>

(e-book in CUHK Lib.)

§ Fourier Series

A function $f: D \to \mathbb{R}$, where $D \subset \mathbb{R}$, is said to be <u>periodic</u> if there exists a nonzero real number r such that for all $x \in D$, we have: $x+r \in D$ and

$$f(x+r) = f(x).$$

r is called a <u>period</u> of f; in that case, assuming that $x-r \in D$ etc, $\pm r, \pm 2r, \cdots, \pm nr$ $(n \in \mathbb{N})$ are periods (of f) also.

<u>Examples</u>

$\boxed{1.}$ Trigonometric functions: sin, cos, tan, etc. are periodic functions (on their respective domains), with period 2π.

$\boxed{2.}$ The functions $\sum_{k=-m}^{n} [a_k \cos(kx) + b_k \sin(kx)]$, where $m, n \in \mathbb{N} \cup \{0\}$, $a_k, b_k \in \mathbb{R}$, are periodic functions (on $\mathbb{R}$) with period 2π. These functions are called <u>trigonometric polynomials</u>.

$\boxed{3.}$ Let $g: [0, 2\pi) \to \mathbb{R}$. Define $f: \mathbb{R} \to \mathbb{R}$ by:

$$f(x+2l\pi) = g(x), \text{ if } x \in [0,2\pi),\ l \in \mathbb{Z} \text{ (the set of integers)}.$$

Then f is a periodic function on $\mathbb{R}$ with period 2π. f is called the <u>2π-periodic extension</u> of g to $\mathbb{R}$.

This procedure can be applied to functions defined on $[-\pi, \pi)$, or $[a,b)$ or $(a,b]$ with $b-a = 2\pi$, or $[a,b]$ if $g(b) = g(a)$ (and $b-a = 2\pi$).

$\boxed{4.}$ Let $h: [0,\pi] \to \mathbb{R}$. Define $g_1: [-\pi, \pi] \to \mathbb{R}$ by

$$g_1(x) = \begin{cases} h(x), & x \in [0,\pi] \\ h(-x), & x \in [-\pi, 0). \end{cases}$$

Then g_1 is <u>even</u>: $g_1(z) = g_1(-z)$, $z \in [-\pi,\pi]$, and $g_1(\pi) = g_1(-\pi)$. So we can apply the procedure in $\underline{3}$ above to g_1, to obtain the 2π-periodic extension f of g_1 to $\mathbb{R}$. Clearly, f is even, i.e. $f(x)=f(-x)$ for all $x \in \mathbb{R}$. f is called the <u>even 2π-periodic extension</u> of h.

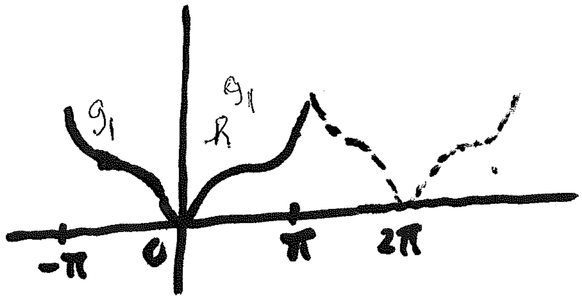
$\boxed{5}$ Let $h: (0,\pi) \to \mathbb{R}$. Let $g_2: [-\pi,\pi) \to \mathbb{R}$ be such that

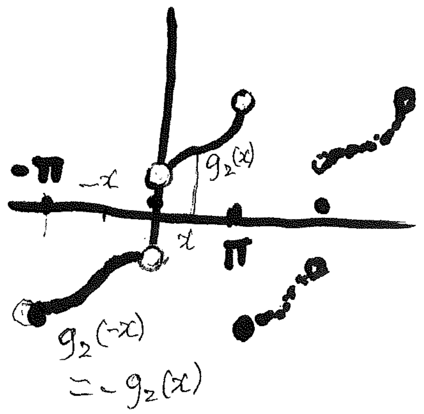
$$g_2(x) = \begin{cases} h(x), & x \in (0,\pi) \\ -h(-x), & x \in (-\pi, 0). \end{cases}$$

[Thus g_2 is odd on $(-\pi,\pi)$.]

$g_2(0)=g_2(-0) \Rightarrow g_2(0)=0$

There are no specific requirement on ~~$g_2(0)$ and~~ $g_2(-\pi)$. Apply to g_2 the procedure in $\underline{3}$ above, to obtain the 2π-periodic extension f of g_2 to $\mathbb{R}$. ~~In general,~~ f is ~~not~~ odd on $\mathbb{R}$.





$\boxed{6}$

Let f be a function defined on $[-\pi,\pi]$ to $\mathbb{R}$, $f: [-\pi,\pi] \to \mathbb{R}$.

f is said to be <u>sectionally continuous</u> if there are $a_1 < a_2 < \cdots < a_n$ in $(-\pi,\pi)$ such that f is continous on $(-\pi, a_1)$, (a_1,a_2), $\cdots$, (a_{n-1}, a_n), (a_n, π) and the one-sided limits $f(-\pi^+)$, $f(a_1^-)$, $f(a_1^+)$, $\cdots$, $f(a_n^-)$, $f(a_n^+)$, $f(\pi^-)$ exist <u>and finite</u>. Then we can defin by

$$\int_{-\pi}^{a_1} f(x)\,dx = \int_{-\pi}^{a_1} f_1(x)\,dx, \quad \text{where } f_1(x) = \begin{cases} f(-\pi^+), & x=-\pi \\ f(x), & x \in (-\pi, a_1) \\ f(a_1^-), & x = a_1 \end{cases}$$

because f_1 is continuous on $[-\pi, a_1]$. We define $\int_{-\pi}^{\pi} f(x)\,dx = \sum_{k=0}^{n} \int_{a_k}^{a_{k+1}} f(x)\,dx$, with $a_0=-\pi$, $a_{n+1}=\pi$.

7 Examples

1. Let $f(x) = |x|$ on $[-\pi, \pi]$. Then f is even and continuous on $[-\pi, \pi]$.

Let $a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx$, $k = 0, 1, 2, \cdots$,

$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx$, $k = 1, 2, \cdots$.

Then by direct calculation / observation,

$b_k = 0$ for $k = 1, 2, \cdots$

$a_k = \frac{2}{\pi} \int_0^{\pi} x \cos(kx)\, dx = \cdots$ (integration by parts)

$$= \begin{cases} \frac{-4}{\pi k^2} & \text{if } k = 1, 3, 5, \cdots \text{ (odd)}, \\ 0 & \text{if } k = 2, 4, 6, \cdots \text{ (even, non-zero)} \\ \pi & \text{if } k = 0 \end{cases}$$

The series $\frac{a_0}{2} + \sum_{k=1}^{\infty} [a_k \cos(kx) + b_k \sin(kx)]$ is called the Fourier series of the function f on $[-\pi, \pi]$.

2. Find the Fourier series of the function x^2 on $[-\pi, \pi]$.

$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos(0 \cdot x) dx = \frac{2}{\pi} \frac{1}{3} x^3 \Big|_0^{\pi} = \frac{2}{\pi} \frac{\pi^3}{3} = \frac{2}{3} \pi^2$.

$a_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos x \, dx = \frac{2}{\pi} \int_0^{\pi} x^2 d(\sin x) = \frac{2}{\pi} \left[x^2 \sin x \Big|_0^{\pi} - \int_0^{\pi} 2x \sin x \, dx \right]$

$= -\frac{2}{\pi} \left[-2x \cos x \Big|_0^{\pi} + 2 \sin x \Big|_0^{\pi} \right] = -4$.

$b_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \sin x \, dx = 0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \sin(kx) \, dx = 0$, $k = 2, 3, \cdots$

? $a_k = \frac{(-1)^k 4}{k^2}$, $k = 2, 3, \ldots$

3. Show that the Fourier series of the function $g(x)=x^2$ on $(0,2\pi)$ is given by

$$\frac{4\pi^2}{3}+\sum_{n=1}^{\infty} 4\left(\frac{\cos nx}{n^2}-\frac{\pi \sin nx}{n}\right).$$

$$a_0=\frac{8\pi^2}{3},\quad a_k=\frac{4}{k^2},\quad k=1,2,\cdots,\quad b_k=-\frac{4\pi}{k},\quad k=1,2,\cdots.$$

<u>Thm 1</u> Suppose f is a periodic function with period 2π and f, f' are sectionally continuous on $[-\pi,\pi]$, then the Fourier series (of f) is convergent. The sum of the Fourier series is equal to $f(x)$ at all numbers x where f is continuous. At the numbers x where f is discontinuous, the sum of the Fourier series is $\frac{1}{2}[f(x^+)+f(x^-)]$.

In short,

$$\frac{a_0}{2}+\sum_{k=1}^{\infty}\left[a_k\cos(kx)+b_k\sin(kx)\right]=\frac{1}{2}\left[\lim_{z\to x^+}f(z)+\lim_{z\to x^-}f(z)\right],$$

for all $x\in(-\pi,\pi)$;

$$\frac{a_0}{2}+\sum_{k=1}^{\infty}\left[a_k\cos(kw)+b_k\sin(kw)\right]=\frac{1}{2}\left[\lim_{z\to\pi^-}f(z)+\lim_{z\to(-\pi)^+}f(z)\right],$$

if $w=-\pi$ or π.

<u>Thm 2</u> Suppose f is sectionally continuous on $[-\pi,\pi]$. Let $\frac{1}{2}a_0+\sum_{k=1}^{\infty}[a_k\cos(kx)+b_k\sin(kx)]$ be the Fourier series of f on $[-\pi,\pi]$. Then for all $\xi<x$ in $[-\pi,\pi]$, we have:

$$\int_\xi^x f(s)\,ds=\int_\xi^x\frac{a_0}{2}\,ds+\sum_{k=1}^{\infty}\left[a_k\int_\xi^x\cos(ks)\,ds+b_k\int_\xi^x\sin(ks)\,ds\right].$$