# Multivariable Calculus

## 1. Functions of several (real) variables

e.g. (1) $F_1 : \mathbb{R}^2 \to \mathbb{R} : \ (x, y) \longmapsto 7x^2y^3$

(2) $F_2 : \mathbb{R}^2 \to \mathbb{R} : \ (x, y) \longmapsto \cos(x+y)$

(3) $F_3 : \mathbb{R}^3 \to \mathbb{R} : \ (x, y, z) \longmapsto \ln(x^2+y^2+z^2+3)$

(4) $F_4 : \{(x,y,z) \in \mathbb{R}^3 : \ x^2+y^2+z^2 \leq 1\} \to \mathbb{R} : \ (x,y,z) \longmapsto \sqrt{2-(x^2+y^2+z^2)}$

## 2. Partial derivatives

Let $F : (a,b)\times(c,d) \to \mathbb{R}$ be a function. $\left[ \begin{array}{l} \text{Recall:} \\ (a,b)\times(c,d) = \left\{ (x,y) : \begin{array}{l} x \in (a,b) \\ y \in (c,d) \end{array} \right\} \end{array} \right]$.

Then $\left.\dfrac{\partial F}{\partial x}\right|_{(s,t)}$, or, $\dfrac{\partial F}{\partial x}(s,t)$, is defined as follows:

$$\frac{\partial F}{\partial x}(s,t) = \lim_{h\to 0} \frac{F(s+h,t)-F(s,t)}{h}$$

where $(s,t) \in (a,b)\times(c,d)$. Thus $\dfrac{\partial F}{\partial x}(s,t)$ is just the derivative at $s$ of the 1-variable function $f : (a,b) \to \mathbb{R} : \ x \longmapsto F(x,t)$ with respect to $x$, i.e.

$$\frac{\partial F}{\partial x}(s,t) = \frac{df}{dx}(s) \left[ = \lim_{h\to 0} \frac{f(s+h)-f(s)}{h} = \lim_{h\to 0} \frac{F(s+h,t)-F(s,t)}{h} \right].$$

Similarly

$$\frac{\partial F}{\partial y}(s,t) \stackrel{\text{def}}{=} \lim_{h\to 0} \frac{F(s,t+h)-F(s,t)}{h}.$$

e.g. For the functions $F_1 - F_4$ in the preceding paragraph:

$$\frac{\partial F_1}{\partial x}(1,2) = \frac{d}{dx}\left[7x^2(2)^3\right]\Big|_{x=1} = 56\left(2x\Big|_{x=1}\right) = 112; \quad \frac{\partial F_2}{\partial x}(1,2) = \frac{d}{dx}\left[\cos(x+2)\right]\Big|_{x=1} = -\sin(1+2);$$

$$\frac{\partial F_1}{\partial y}(1,2) = \frac{d}{dy}\left[7(1^2)y^3\right]\Big|_{y=2} = 7\left(3y^2\Big|_{y=2}\right) = 7(3)(2^2) = 84; \quad \frac{\partial F_2}{\partial y}(1,2) = \frac{d}{dy}\left[\cos(1+y)\right]\Big|_{y=2} = -\sin 3.$$

Because $\frac{\partial F}{\partial x}$ is itself a function of severable variables, it has its also partial derivatives. We denote

$$\frac{\partial^2 F}{\partial x^2} \stackrel{\text{def}}{=} \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial x}\right), \quad \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right), \quad \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right), \text{ etc.}$$

For example,

$$\frac{\partial^2 F_1}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial x}\right) = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}(7x^2y^3)\right] = \frac{\partial}{\partial y}\left[14xy^3\right] = 42xy^2;$$

$$\frac{\partial^2 F_1}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial y}\right) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}(7x^2y^3)\right] = \frac{\partial}{\partial x}\left[21x^2y^2\right] = 42xy^2;$$

$$\frac{\partial^2 F_1}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial x}\right) = \frac{\partial}{\partial x}\left[14xy^3\right] = 14y^3;$$

$$\frac{\partial^2 F_1}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial y}\right) = \frac{\partial}{\partial y}\left[21x^2y^2\right] = 42x^2y;$$

$$\frac{\partial^3 F_1}{\partial y^3} = \frac{\partial}{\partial y}\left(\frac{\partial^2 F_1}{\partial y^2}\right) = \frac{\partial}{\partial y}\left[42x^2y\right] = 42x^2, \text{ etc.}$$

We notice that $\frac{\partial^2 F_1}{\partial x \partial y} = \frac{\partial^2 F_1}{\partial y \partial x}$ (thus, in this case, the order of differentiation does not matter). Under fairly general conditions, we can prove that

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x},$$

$$\frac{\partial^3 F}{\partial x \partial y \partial x} = \frac{\partial^3 F}{\partial y \partial x^2} = \frac{\partial^3 F}{\partial x^2 \partial y} \quad \text{etc.}$$

However, there are examples where

$$\frac{\partial^2 F}{\partial x \partial y} \neq \frac{\partial^2 F}{\partial y \partial x}.$$

Read more examples given in the related section(s) of the course lecture notes (by Dr. Jeff Wang).

<u>Chain rules</u>

Suppose $f: \mathbb{R}^2 \to \mathbb{R} : (x, y) \mapsto f(x,y)$, and $\phi: \mathbb{R}^2 \to \mathbb{R}$, $\psi: \mathbb{R}^2 \to \mathbb{R}$. Then a function $F: \mathbb{R}^2 \to \mathbb{R}$ is given by:

$$F(u,v) = f(\phi(u,v), \psi(u,v)),$$

e.g. if $f(x,y) = \cos(3x+7y)$, $\phi(u,v) = uv$, $\psi(u,v) = u-v$, then

$$F(u,v) = \cos[3(uv) + 7(u-v)]. \quad \cdots (*)$$

The partial derivative $\frac{\partial F}{\partial u}$ can be obtained as follows:

$$\frac{\partial F}{\partial u} = \left.\frac{\partial f}{\partial x}\right|_{\substack{x=\phi(u,v)\\ y=\psi(u,v)}} \frac{\partial \phi}{\partial u} + \left.\frac{\partial f}{\partial y}\right|_{\substack{y=\psi(u,v)\\ x=\phi(u,v)}} \frac{\partial \psi}{\partial u} \quad \cdots (1)$$

Formula (1) is called a <u>chain rule.</u> E.g. For the function $F$ given by $(*)$:

$$\frac{\partial F}{\partial u} = \frac{\partial[\cos(3x+7y)]}{\partial x} \frac{\partial (uv)}{\partial u} + \frac{\partial[\cos(3x+7y)]}{\partial y} \frac{\partial (u-v)}{\partial u}$$

$$= -3\sin(3x+7y)\Big|_{\substack{x=uv\\ y=u-v}} v + -7\sin(3x+7y)\Big|_{\substack{x=uv\\ y=u-v}} (1)$$

$$= -3v\sin(3uv+7u-7v) - 7\sin(3uv+7u-7v).$$

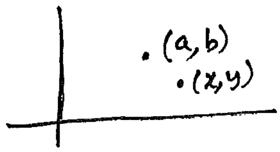
Similarly we have

$$\frac{\partial F}{\partial v} = \left.\frac{\partial f}{\partial x}\right|_{\substack{x=\phi(u,v)\\ y=\psi(u,v)}} \frac{\partial \phi}{\partial v} + \left.\frac{\partial f}{\partial y}\right|_{\substack{x=\phi(u,v)\\ y=\psi(u,v)}} \frac{\partial \psi}{\partial v} \quad \cdots (2)$$

Formula (2) is also called a <u>chain rule.</u> Formulas (1) & (2) hold under very general conditions. There are further generalizations. (cf. Lecture Notes by Dr. T. Wong)

## 3. Continuous functions

Let $F: \mathbb{R}^2 \to \mathbb{R}$, and let $(a,b) \in \mathbb{R}^2$. Then $F$ is said to be continuous at $(a,b)$ if $|F(x,y) - F(a,b)|$ is very small whenever $|x-a|$ and $|y-b|$ are very small. Intuitively, this means $F(x,y)$ is very close to $F(a,b)$ whenever $x$ is very close to $a$ and $y$ is very close to $b$. [The last two clauses mean that the point $(x,y)$ is very close to the point $(a,b)$, as the distance between $(x,y)$ and $(a,b)$ is defined to be $\sqrt{(x-a)^2+(y-b)^2}$.]




e.g. the functions $F_1 \cdots F_4$ are continuous at every point of their domains:

(1) $F_1$ is continuous at each $(a,b) \in \mathbb{R}^2$, because

$$7x^2y^3 - 7a^2b^3 = 7(x^2-a^2)y^3 + 7a^2(y^3-b^3)$$
$$\approx 7(0)b^3 + 7a^2(0) = 0$$

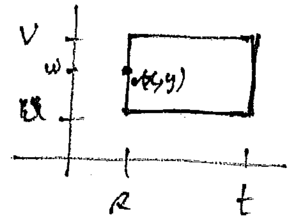
whenever $x-a \approx 0$ and $y-b \approx 0$, i.e. $|7x^2y^3 - 7a^2b^3|$ is very small whenever $|x-a|$ and $|y-b|$ are very small.

(2) $F_2$ is continuous at each $(a,b) \in \mathbb{R}^2$, because:

when ($x-a \approx 0$ and $y-b \approx 0$, we have $x+y \approx a+b$, and therefore
$$\cos(x+y) \approx \cos(a+b).$$

[i.e. $\cos(x+y)$ is very close to $\cos(a+b)$ whenever $x$ is close to $a$ and $y$ is close to $b$]

Let $F: [s,t]\times[u,v] \to \mathbb{R}$, and $w \in [u,v]$. Then $F$ is said to be continuous at $(s,w)$ if $F(x,y) - F(s,w)$ is close to zero, whenever $(x,y)$ is close to $(s,w)$ (i.e. $x$ close to $s$, and $y$ close to $w$) and $(x,y) \in [s,t]\times[u,v]$.




Similar consideration is applied to points on the other edges of the rectangle $[s,t]\times[u,v]$: $(t,w)$, $(r,u)$, $(r,v)$ (where $r \in [s,t]$).

Let $F: D \to \mathbb{R}$ where $D \subset \mathbb{R}^2$ (or $D \subset \mathbb{R}^3$) is a rectangle/disk with/without edges (or, cube/ball with/without boundary). Then $F$ is said to be continuous on $D$ if $F$ is continuous at every point of $D$.

e.g. $F_5$ is continuous on $D = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \le 1\}$, where

$$F_5 : D \to \mathbb{R} : (x,y,z) \mapsto \sqrt{2-(x^2+y^2+z^2)}.$$

Thm Let $f: [s,t] \to \mathbb{R}$ be continuous, and let $F: [s,t]\times[u,v] \to \mathbb{R}$ be defined by:

$$F(x,y) \overset{\text{def}}{=} f(x), \quad x \in [s,t],\ y \in [u,v].$$

Then $F$ is continuous on $[s,t]\times[u,v]$.

Similarly the function $G: [u,v]\times[s,t] \to \mathbb{R} : (x,y) \mapsto f(y)$ is continuous on $[u,v]\times[s,t]$.

Many theorems of continuous functions of 1-variable generalize to the cases of continuous functions of several variables, though sometimes in amended form.

## 4. Iterated Integrals & Double Integrals

Consider the function $F: \mathbb{R}^2 \to \mathbb{R}: (x,y) \mapsto \sin(x+3y)$. Then

$$\int_2^5 F(x,y)\,dx \quad \left[= \int_2^5 \sin(x+3y)\,dx = -\cos(x+3y)\Big|_{x=2}^{x=5}\right]$$

is a continuous function on $[1,4]$. Let us call this function $g$, i.e.

$$g: [1,4] \to \mathbb{R}: y \mapsto g(y) = -\cos(x+3y)\Big|_{x=2}^{x=5} = \cos(2+3y) - \cos(5+3y)$$

$g$ is continuous on $[1,4]$, and we can consider the integral $\int_1^4 g(y)\,dy$. This integral is usually written as

$$\int_1^4 \left[\int_2^5 F(x,y)\,dx\right] dy, \quad \text{or simply} \quad \int_1^4 \int_1^5 F(x,y)\,dx\,dy,$$

$$\iint_{1\le y\le 4,\ 1\le x\le 5} F(x,y)\,dx\,dy$$

and is called an <u>iterated integral</u>.

Similarly we define $\displaystyle\int_2^5 \left[\int_1^4 F(x,y)\,dy\right] dx$, or simply $\displaystyle\int_2^5 \int_1^4 F(x,y)\,dy\,dx$

One can easily check that

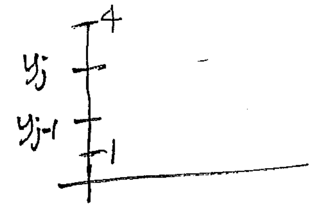
$$\int_2^5 \int_1^4 F(x,y)\,dy\,dx = \int_1^4 \int_2^5 F(x,y)\,dx\,dy.$$

$$\int_2^5 \int_1^4 F(x,y)\,dy\,dx = \frac{1}{3}\int_2^5 \left[\cos(x+3) - \cos(x+12)\right] dx = \frac{1}{3}\left[\sin(5+3) - \sin(2+3) + \sin(2+12) - \sin(5+12)\right]$$

$$\int_1^4 \int_2^5 F(x,y)\,dx\,dy = \int_1^4 \left[\cos(2+3y) - \cos(5+3y)\right] dy = \frac{1}{3}\left[\sin(5+3) - \sin(5+12) + \sin(2+12) - \sin(2+3)\right]$$

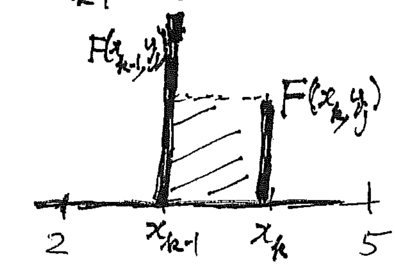
Is this merely accidental? No, it is not accidental. It is true under <u>fairly general conditions</u>

Let us see why this is the case. Recall the definition of definite integral. $\int_1^4 \left[ \int_2^5 F(x,y)\,dx \right] dy$ is (a kind of) limit of certain sums: let $P:\ 1 = y_0 < y_1 < \cdots < y_{j-1} < y_j < \cdots < y_{n-1} < y_n = 4$ be a partition of $[1,4]$, then for $\|P\|$ small enough,




$$\int_1^4 \left[ \int_2^5 F(x,y)\,dx \right] dy \approx \sum_{j=1}^{n} g(y_j)\,\Delta y_j\,,$$

where $\Delta y_j \stackrel{\text{def}}{=} y_j - y_{j-1}$, and $g(y_j) = \int_2^5 F(x,y_j)\,dx$. But $\int_2^5 F(x,y_j)\,dx$ is also (a kind of) limit of certain sums: Let $Q:\ 2 = x_0 < x_1 < \cdots < x_{k-1} < x_k < \cdots x_{\ell-1} < x_\ell = 5$ be a partition of $[2,5]$, then for $\|Q\|$ small enough,




$$g(y_j) = \int_2^5 F(x,y_j)\,dx \approx \sum_{k=1}^{\ell} F(x_k,y_j)\,\Delta x_k\,,$$

where $\Delta x_k \stackrel{\text{def}}{=} x_k - x_{k-1}$. Thus

$$\int_1^4 \left[ \int_2^5 F(x,y)\,dx \right] dy \approx \sum_{j=1}^{n} \left[ \sum_{k=1}^{\ell} F(x_k,y_j)\,\Delta x_k \right] \Delta y_j$$

$$= \sum_{\substack{1\le j\le n \\ 1\le k\le \ell}} F(x_k,y_j)\,\Delta x_k\,\Delta y_j = \sum_{k=1}^{\ell} \left[ \sum_{j=1}^{n} F(x_k,y_j)\,\Delta y_j \right] \Delta x_k \approx \int_2^5 \left[ \int_1^4 F(x,y)\,dy \right] dx$$

Now $\Delta x_k \Delta y_j$ = area of the rectangle $[x_{k-1},x_k] \times [y_{j-1},y_j] \stackrel{\text{def}}{=} R_{k,j}$, and $R_{k,j}$, $1\le k\le \ell$, $1\le j\le n$, form a partition of the rectangle $[2,5]\times[1,4]$ in an obvious sense (similar to $[x_{k-1},x_k]$, $k=1,2,\cdots,\ell$, form a partition of $[2,5]$). Therefore we can anticipate that for our continuous function $F$, with $\|P\|$, $\|Q\|$ small enough, the sum (similar to the Left-Riemann sum for a function of 1-variable) $\sum_{\substack{1\le j\le n \\ 1\le k\le \ell}} F(x_k,y_j)\,\Delta x_k\,\Delta y_j$ converges to (clusters about) a number (similar to the definite integral of a function of 1-variable over an interval $[a,b]$): $\iint_{[2,5]\times[1,4]} F(x,y)\,dx\,dy$, also denoted as $\iint_{[2,5]\times[1,4]} F(x,y)\,dA$ by some authors.

(**Our basic assumption here is: $F$ is continuous on $[2,5]\times[1,4]$**)

i.e. $\displaystyle\sum_{\substack{1\le j\le n\\ 1\le k\le l}} F(x_k, y_j)\,\Delta x_k\,\Delta y_j \approx \iint_{[2,5]\times[1,4]} F(x,y)\,dx\,dy$.

The number $\displaystyle\iint_{[2,5]\times[1,4]} F(x,y)\,dx\,dy$ is called the <u>double integral</u> of F over $[2,5]\times[1,4]$. And the above observation suggests that

$$\int_1^4\left[\int_2^5 F(x,y)\,dx\right]dy = \int_2^5\left[\int_1^4 F(x,y)\,dy\right]dx = \iint_{[2,5]\times[1,4]} F(x,y)\,dx\,dy.$$

Actually the same argument applies to all continuous functions $G(x,y)$ on a rectangle $[a,b]\times[c,d]$ in $\mathbb{R}^2$, and we have equalities similar to the above.

How about $\displaystyle\int_2^{10-2y} F(x,y)\,dx$ for $y\in[1,4]$?

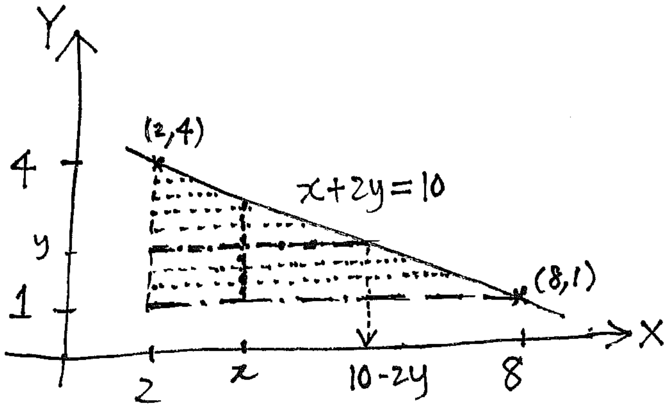
It is still a continuous function of $y$ (on $[1,4]$). So, we can consider

$$\int_1^4\left[\int_2^{10-2y} F(x,y)\,dx\right]dy.$$

Is it equal to $\displaystyle\int_2^{10-2y}\left[\int_1^4 F(x,y)\,dy\right]dx$? Obviously, that is quite unlikely: the first iterated integral is a number, but the second iterated integral is a function of $y$.
How can we interchange the order of integration? Well, let us consider the double integral

$$\iint_{\substack{2\le x\le 10-2y\\ 1\le y\le 4}} F(x,y)\,dx\,dy.$$

The domain of integration $\{(x,y)\in\mathbb{R}^2 : 2\le x\le 10-2y,\ 1\le y\le 4\}$ can be illustrated in the following figure:




The domain can be described as

$$\left\{(x,y)\in\mathbb{R}^2 : 1\le y\le \tfrac{1}{2}(10-x),\ \ 2\le x\le 8\right\}$$

Therefore

$$\iint\limits_{\substack{2\le x\le 10-2y\\ 1\le y\le 4}} F(x,y)\,dx\,dy = \iint\limits_{\substack{1\le y\le \frac{1}{2}(10-x)\\ 2\le x\le 8}} F(x,y)\,dx\,dy$$

$$= \int_2^8 \left[\int_1^{5-\frac{x}{2}} F(x,y)\,dy\right] dx.$$

Thus we have:

$$\int_1^4 \left[\int_2^{10-2y} F(x,y)\,dx\right] dy = \int_2^8 \left[\int_1^{5-\frac{x}{2}} F(x,y)\,dy\right] dx.$$

This last equality can be confirmed by direct calculation.

$$\text{R.H.S.} = \int_1^4\left[\int_2^{10-2y} \sin(x+3y)\,dx\right] dy = \int_1^4 \left[\cos(2+3y) - \cos(10+y)\right] dy$$

$$= \frac{1}{3}\sin(2+3y)\Big|_1^4 - \sin(10+y)\Big|_1^4 = \frac{\sin(14)}{3} - \frac{\sin(5)}{3} + \sin(11) - \sin(14)$$

$$= \sin(11) - \frac{\sin(5)}{3} - \frac{2}{3}\sin(14)$$

$$\text{L.H.S.} = \int_2^8 \left[\int_1^{5-\frac{x}{2}} \sin(x+3y)\,dy\right] dx = \int_2^8 \frac{1}{3}\left[\cos(x+3) - \cos\left(-\frac{x}{2}+15\right)\right] dx$$

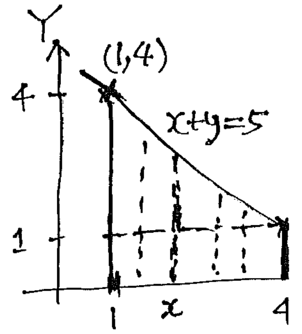
$$= \frac{1}{3}\left[\sin(x+3)\Big|_2^8 + 2\sin\left(-\frac{x}{2}+15\right)\Big|_2^8\right] = \frac{1}{3}\left[\sin(11) - \sin(5) + 2\sin(11) - 2\sin(14)\right]$$

$$= \sin(11) - \frac{1}{3}\sin(5) - \frac{2}{3}\sin(14).$$

As another example, consider

$$\int_1^4 \left[ \int_0^{5-x} \cos(x+y)\,dy \right] dx$$

$$= \int_1^4 \left[ \int_1^{5-y} \cos(x+y)\,dx \right] dy + \int_0^1 \left[ \int_1^4 \cos(x+y)\,dx \right] dy,$$




$\{(x,y)\in\mathbb{R}^2 : 0\le y\le 5-x,\ 1\le x\le 4\}$

$= \{(x,y)\in\mathbb{R}^2 : 1\le x\le 5-y,\ 1\le y\le 4\}$

$\cup \underbrace{\{(x,y)\in\mathbb{R}^2 : 1\le x\le 4,\ 0\le y\le 1\}}_{[1,4]\times[0,1]}$

which is confirmed by direct calculation:

$$\text{L.H.S.} = \int_1^4 \sin(x+y)\Big|_{y=0}^{y=5-x} dx$$

$$= \int_1^4 [\sin(5) - \sin(x)]\,dx = 3\sin(5) + \cos(4) - \cos(1),$$

$$\text{R.H.S.} = \int_1^4 \sin(x+y)\Big|_{x=1}^{x=5-y} dy + \int_0^1 \sin(x+y)\Big|_{x=1}^{x=4} dy$$

$$= \int_1^4 [\sin(5) - \sin(1+y)]\,dy + \int_0^1 [\sin(4+y) - \sin(1+y)]\,dy$$

$$= 3\sin(5) + \cos(1+y)\Big|_{y=1}^{y=4} + \cos(1+y)\Big|_{y=0}^{1} - \cos(4+y)\Big|_{y=0}^{1}$$

$$= 3\sin(5) + \cos(5) - \cos(2) + \cos(2) - \cos(1) - \cos(5) + \cos(4)$$

$$= 3\sin(5) + \cos(4) - \cos(1).$$

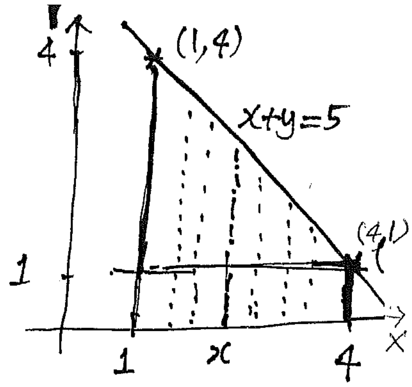
The above discussion in §§1-4 can all be carried out for more than 2 variables.

As a further example, we may consider

$$\int_1^4 \left[ \int_0^{5-x} x^2 y^3 \, dy \right] dx$$

$$= \int_1^4 \left[ \int_1^{5-y} x^2 y^3 \, dx \right] dy + \int_0^1 \left[ \int_1^4 x^2 y^3 dx \right] dy,$$

as




$$\{(x,y) \in \mathbb{R}^2 : 0 \le y \le 5-x, \ 1 \le x \le 4\}$$
$$= \{(x,y) \in \mathbb{R}^2 : 1 \le x \le 5-y, \ 1 \le y \le 4\}$$
$$\cup \{(x,y) \in \mathbb{R}^2 : 1 \le x \le 4, \ 0 \le y \le 1\}$$

which is confirmed by direct calculation:

$$\text{L.H.S.} = \int_1^4 \left[\frac{1}{4} x^2 y^4\right]\Big|_{y=0}^{y=5-x} dx$$

$$= \int_1^4 \frac{1}{4}\left[x^2 (5-x)^4\right] dx = \frac{1}{4} \int_1^4 \left( x^6 - 20x^5 + 150x^4 - 500x^3 + 625x^2 \right) dx$$

$$= \frac{1}{4}\left[\frac{1}{7}(4^7-1) - \frac{10}{3}(4^6-1) + 30(4^5-1) - 125(4^4-1) + \frac{625}{3}(4^3-1)\right]$$

$$= \frac{1}{4}\left[\frac{4^7}{7} - \frac{10(4^6)}{3} + 30(4^5) - 125(4^4) + \frac{625}{3}(4^3) - \frac{771}{7}\right] = 157.607$$

$$\text{R.H.S.} = \int_1^4 \left[\frac{1}{3} x^3 y^3\right]\Big|_{x=1}^{x=5-y} dy + \int_0^1 \frac{y^3}{3}(4^3-1) \, dy$$

$$= \int_1^4 \frac{y^3}{3}\left[(5-y)^3 - 1\right] dy + \frac{21}{4}$$

$$= \frac{1}{3} \int_1^4 \left(124y^3 - 75y^4 + 15y^5 - y^6\right) dy + \frac{21}{4}$$

$$= \frac{1}{3}\left[\frac{124}{4}(4^4-1) - 15(4^5-1) + \frac{5}{2}(4^6-1) - \frac{1}{7}(4^7-1)\right] + \frac{21}{4}$$

$$= 157.607$$