§ Differentiation

Let $f: (a,b) \longrightarrow \mathbb{R}$, and $c \in (a,b)$. Suppose

$$g(h) = \frac{f(c+h) - f(c)}{h}$$

(this makes sense if $c+h \in (a,b)$ & $h \neq 0$
i.e. if $a-c < h < b-c$, & $h \neq 0$
i.e. if $h \in (-(c-a), b-c) \setminus \{0\}$)

so $g: (-(c-a),\ b-c) \setminus \{0\} \to \mathbb{R}$.

If $\lim_{h \to 0^-} g(h) = l \in \mathbb{R} \cup \{-\infty, \infty\}$, i.e. if

$$\lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} = l \text{ (a real number or } \pm\infty\text{)},$$

then we say: "f has a left-derivative at c, which has the value l,"

& write: "$f'_-(c) = l$" $\quad Lf'(c)$

In short, $\displaystyle f'_-(c) = \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h}$,

in other words, $\displaystyle f'_-(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c}$,

provided the limit (on the right side of $=$) exists (as a real number, or $-\infty$, or ∞).

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Ex. Let
$$f(x) = \begin{cases} -x, & x > 3, \\ 9, & x = 3, \\ x^2, & x < 3, \end{cases}$$
and $c = 3$.

Then for $h < 0$,
$$\frac{f(3+h) - f(3)}{h} = \frac{(3+h)^2 - 9}{h}$$
$$= \frac{6h + h^2}{h} = 6 + h,$$

hence
$$\lim_{h \to 0^-} \frac{f(3+h) - f(3)}{h} = \lim_{h \to 0^-} (6 + h)$$
$$= 6.$$

Thus $f'_-(3) = 6$.

Rf'(c)

Similarly we define
$$f'_+(c) \overset{\text{def}}{=\!=} \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h}$$

provided the limit (on the right of =) exists/makes sense.

For the same f in the above example,
$$\lim_{h \to 0^+} \frac{f(3+h) - f(3)}{h} = \lim_{h \to 0^+} \frac{-(3+h) - 9}{h} = \lim_{\substack{h \to 0 \\ 0^+}} \left(-1 - \frac{12}{h}\right)$$
$$= -\infty,$$

hence $f'_+(3) = -\infty$.

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When $f'_-(c) = l = f'_+(c)$, and $l \in \mathbb{R}$ we say:

"f has a derivative at c, which has the value l", we write $f'(c) = l$

"f is <u>differentiable</u> at c if $f'(c) =$ a real number",

and write: "$f'(c) = l$",

or "$\frac{df}{dx}(c) = l$", "$\left.\frac{df}{dx}\right|_{x=c} = l$"

<u>Examples</u>

<u>1.</u> Let $f(x) = x$. Then $f : \mathbb{R} \to \mathbb{R}$, and for every $x \in \mathbb{R}$,

$$\lim_{h\to 0^-} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0^-} \frac{(x+h) - x}{h} = \lim_{h\to 0^-} 1$$

$= 1$, i.e. $f'_-(x) = 1$.

Similarly, $f'_+(x) = 1$. Thus

$f'(x) = 1$ for every $x \in \mathbb{R}$.

I.e. $(x)' = 1$, or $\frac{d(x)}{dx} = 1$.

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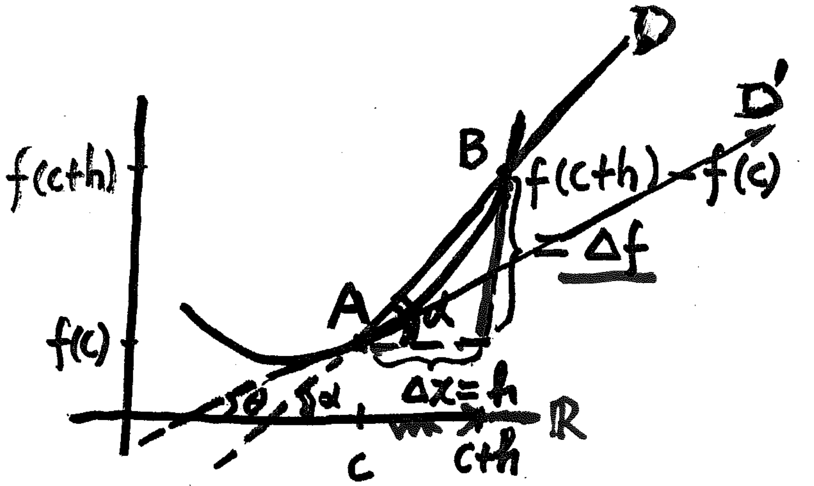
§ Derivatives and tangents

Consider the graph of a function $f: (a,b) \longrightarrow \mathbb{R}$, and let $c \in (a,b)$.



$B = (c+h, f(c+h))$

$A = (c, f(c))$

$$\frac{\Delta f}{\Delta x} = \text{tg}\,\alpha$$

$$\downarrow$$

$$f'(c)$$

When B moves along the graph of f, and approaches A, the ~~segment~~ line ABD tends to a limitting position (segment) AD', which has a slope of $f'(c)$ [i.e. $f'(c) = \text{tg}$ (angle btn AD' & the positive x-axis)] $= \text{tg}\,\theta$

the **tangent** to the graph of f at the point $(c, f(c))$

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~~III-12~~

The ratio/fraction $\dfrac{f(c+h)-f(c)}{h}$ has the following interpretation:

the numerator $f(c+h)-f(c)$

$=$ change in the function value,

$= \Delta f$

the denominator $h = (c+h)-c$

$=$ change in the independent variable

$= \Delta x$

the fraction $= \dfrac{\Delta f}{\Delta x}$ = ratio of Δf by Δx

= change in f per unit change in x.

rate of change

(from the left side)

$$\boxed{f'_-(c) = \lim_{\substack{\Delta x \to 0 \\ \Delta x < 0}} \frac{\Delta f}{\Delta x}}$$

Suppose $f'_-(c) \in \mathbb{R}$. Then

$f(c) + f'_-(c) \cdot h$

good approx.

$\approx f(c+h)$ when $h<0$ & $|h|$ is very small

because $\dfrac{f(c+h)-f(c)}{h} \to f'_-(c)$ as $h \to 0^-$

2. Let $g(x) = \sin x$. Then $g: \mathbb{R} \to \mathbb{R}$, and for every $x \in \mathbb{R}$,

$$\frac{g(x+h)-g(x)}{h} = \frac{1}{h}\left[\sin(x+h) - \sin(x)\right]$$

$$= \frac{2}{h}\left[\cos\left(x+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)\right] = \cos\left(x+\frac{h}{2}\right)\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \to \cos x \text{ as } h \to 0$$

$$= \frac{1}{h}\left[\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)\right]$$

$$= \cos(x)\cdot\frac{\sin h}{h} + \sin(x)\cdot\frac{\cos(h)-1}{h}$$

$$= \cos(x)\frac{\sin h}{h} + \sin(x)\frac{-2\sin^2\left(\frac{h}{2}\right)}{h}$$

$$= \cos(x)\frac{\sin h}{h} - \sin(x)\cdot\left[\frac{\sin(h/2)}{h/2}\right]^2\cdot\frac{h}{2}$$

$$\therefore \lim_{h\to 0^-}\frac{g(x+h)-g(x)}{h} = \left[\lim_{h\to 0^-}\frac{\sin(h)}{h}\right]\cos(x) - \sin(x)\left[\lim_{h\to 0^-}\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right]^2\cdot\lim_{h\to 0^-}\left(\frac{h}{2}\right)$$

$$= \cos(x),$$

i.e. $g'_-(x) = \cos x$.

Similarly, we have: $g'_+(x) = \cos x$.

Thus, $g'(x) = \cos x$, for every $x \in \mathbb{R}$.

I.e. $\underline{(\sin x)' = \cos x}$, $\underline{\frac{d}{dx}(\sin x) = \cos x}$

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We also have

$$\frac{d}{dx}(e^x) = e^x, \quad \frac{d}{dy}(\ln y) = \frac{1}{y},$$

$$\frac{d}{dx}(b^x) = (\ln b)\, b^x, \quad \frac{d}{dy}(\log_b y) = \frac{1}{(\ln b)\, y}$$

where $b > 0$ (and $\neq 1$) is a constant.

Pf Suppose $h > 0$. Recall that

$$e^h = \lim_{n\to\infty} \left(1 + \frac{h}{n}\right)^n,$$

hence $$\frac{e^h - 1}{h} = \lim_{n\to\infty} \left[\frac{(1+\frac{h}{n})^n - 1}{h}\right]$$

Now for $n \geq 2$,

$$\frac{1}{h}\left[\left(1+\frac{h}{n}\right)^n - 1\right] = 1 + \frac{h}{2}\,\frac{n}{n}\left(\frac{n-1}{n}\right) + \underbrace{\frac{h^2}{6}\,\frac{n}{n}\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right) + \cdots}_{\text{positive terms}}$$

$$\therefore\ 1 + \frac{h}{2}\left(1 - \frac{1}{n}\right) \leq \frac{1}{h}\left[\left(1+\frac{h}{n}\right)^n - 1\right] \leq 1 + \frac{h}{2} + \left(\frac{h}{2}\right)^2 + \cdots$$

$$\leq 1 + \left[\frac{h}{2} - \left(\frac{h}{2}\right)^{n+1}\right] \Big/ \left(1 - \frac{h}{2}\right),$$

$$\therefore\ 1 + \frac{h}{2} \leq \frac{e^h - 1}{h} = \lim_{n\to\infty} \frac{1}{h}\left[\left(1+\frac{h}{n}\right)^n - 1\right] \leq 1 + \frac{h}{2} \Big/ \left(1 - \frac{h}{2}\right), \text{ if } \frac{h}{2} < 1 \text{ also}$$

$$\therefore\ \lim_{h\to 0^+} \frac{e^h - 1}{h} = 1.$$ Similarly we have $$\lim_{h\to 0^-} \frac{e^h - 1}{h} = 1.$$

Therefore $$\lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = e^x \lim_{h\to 0} \frac{e^h - 1}{h} = e^x.$$

The other formulas follow readily.

§ Higher Derivatives

$f''(x)$, $\frac{d^2f(x)}{dx^2}$, $\frac{d}{dx}\left(\frac{df(x)}{dx}\right)$ are different notations for the derivative of the function $f'(x)$ (i.e. $\frac{df(x)}{dx}$).

Example $(x)'' = [(x)']' = [1]' = 0$.

§ Basic Theorems on Derivatives

→ **Thm 1** If f is differentiable at c, then f is continuous at c.

Thm 2 Let $f: (a,b) \to \mathbb{R}$, $g: (a,b) \to \mathbb{R}$, and $c \in (a,b)$. Suppose $f'_-(c)$, $g'_-(c)$ are real numbers.

Then (i) $(f+g)'_-(c) = f'_-(c) + g'_-(c)$

(ii) $(fg)'_-(c) = f'_-(c) \cdot g(c) + f(c) \cdot g'_-(c)$

(iii) $$\left(\frac{f}{g}\right)'_-(c) = \frac{g(c)\, f'_-(c) - f(c)\, g'_-(c)}{[g(c)]^2},$$

provided $g(c) \neq 0$.

[The tag "$-$" can be replaced by "$+$" throughout, or dropped throughout.]

Thm 3 (i) Let $a_n, a_{n-1}, \cdots, a_0$ be real numbers.

Then $(a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0)'$

$= n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \cdots + a_1$.

(ii) $(\cos x)' = -\sin x$, $(\tan x)' = \sec^2 x$,

$(\operatorname{ctg} x)' = -\csc^2 x$, $(\sec x)' = (\tan x)(\sec x)$, $(\csc x)' = -(\operatorname{ctg} x)(\csc x)$.

$\dfrac{f(c+h)-f(c)}{h}$ is close to $f'(c)$ when h is close to 0

$$\frac{f(c+h)-f(c)}{h} \approx f'(c)$$

$$\left|f(c+h)-f(c)\right| \approx |h|\,\underbrace{\left|f'(c)\right|}_{\text{a fixed no.}} < \varepsilon$$ whenever $|h| < \delta$

$\delta = \dfrac{\varepsilon}{|f'(c)|}$ if $f'(c) \neq 0$

Thm 4 Suppose $f:(a,b)\to\mathbb{R}$, $g:(c,d)\to\mathbb{R}$, $x\in(a,b)$, $y=f(x)$, $f((a,b))\subset(c,d)$. Suppose $f'(x)$, $g'(y)$ are real numbers. Then

$f(x)\in(c,d)$ whenever $x\in(a,b)$

$$(g\circ f)'(x) = g'(y)f'(x) = g'(f(x))f'(x).$$

[This is known as the <u>chain rule</u>.]

<u>Sketch of proof</u> For $|h|$ suff. small,

$$\begin{aligned}
&(g\circ f)(x+h)-(g\circ f)(x)\\
&= g(f(x+h)) - g(f(x))\\
&\approx g'(f(x))\,[f(x+h)-f(x)] \quad (\text{as } f \text{ is continuous at } x)\\
&\approx g'(f(x))\,[f'(x)\,h],
\end{aligned}$$

$\therefore$ for $h\neq 0$ (suff. small $|h|$),

$$\frac{g\circ f\,(x+h)-g\circ f\,(x)}{h}\approx g'(f(x))\,f'(x).$$

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(1) $x \neq x' \Rightarrow f(x) \neq f(x')$

Thm 5 Suppose $f : (a,b) \to (c,d)$ is bijective,

(2) for any $y \in (c,d)$, $y = f(x)$ for some $x \in (a,b)$

$x_0 \in (a,b)$, and f is differentiable at x_0, with $f'(x_0) \neq 0$. Then the inverse map f^{-1}: $(c,d) \to (a,b)$ is differentiable at $y_0 \overset{\text{def}}{=} f(x_0)$, and

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)} .$$

This is called an inverse function theorem.

Example $2\frac{1}{2}$. $\frac{d(\ln x)}{dx} = \frac{1}{x}$ for $x > 0$. [Revisit p. IV 19 i.e. p. V1.]

$y = \ln x$, $e^y = x$, $\ln x$ is the inverse of $\underset{\displaystyle f(y)}{\exp(y)}$

$$\frac{d(\ln x)}{dx} = \frac{d(f^{-1}(x))}{dx} = \frac{1}{\frac{d f(y)}{dy}} = \frac{1}{e^y} = \frac{1}{x}$$

Ex. 3. Let $\arcsin : (-1,1) \to (-\frac{\pi}{2}, \frac{\pi}{2})$ be the inverse of $\sin : (-\frac{\pi}{2}, \frac{\pi}{2}) \to (-1,1)$

$$\frac{d}{dy}(\arcsin y) = \frac{1}{\frac{d}{dx}(\sin x)} = \frac{1}{\cos x} = \frac{1}{\pm\sqrt{1-\sin^2 x}}$$

$\cos^2 x + \sin^2 x = 1$

$$= \cancel{\pm}\frac{1}{\sqrt{1-y^2}}, \quad y \in (-1,1)$$

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Ex.4. $$\frac{d}{dx}\left[\sin(x^2)\right] = \left.\frac{d \sin y}{dy}\right|_{y=x^2} \frac{d(x^2)}{dx}$$

$$= \left.\cos(y)\right|_{y=x^2} (2x) = 2x\cos(x^2), \text{ for all } x \in \mathbb{R}.$$

(1)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\sin\left(\frac{x}{\cos x}\right) &= \frac{\mathrm{d}\sin u}{\mathrm{d}x} && \text{where } u = \tfrac{x}{\cos x}\\
&= \frac{\mathrm{d}\sin u}{\mathrm{d}u}\cdot\frac{\mathrm{d}u}{\mathrm{d}x}\\
&= \cos\left(\frac{x}{\cos x}\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{\cos x}\right)\\
&= \cos\left(\frac{x}{\cos x}\right)\frac{(\cos x)(x)' - x(\cos x)'}{\cos^2 x}\\
&= \cos\left(\frac{x}{\cos x}\right)\frac{\cos x + x\sin x}{\cos^2 x}.
\end{aligned}$$

(2)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x}\sqrt[3]{1+x^2} &= \frac{\mathrm{d}\sqrt{x}}{\mathrm{d}x}\sqrt[3]{1+x^2} + \sqrt{x}\frac{\mathrm{d}\sqrt[3]{1+x^2}}{\mathrm{d}x}\\
&= \frac{1}{2\sqrt{x}}\sqrt[3]{1+x^2} + \sqrt{x}\frac{\mathrm{d}\sqrt[3]{u}}{\mathrm{d}x} && \text{where } u = 1+x^2\\
&= \frac{1}{2\sqrt{x}}\sqrt[3]{1+x^2} + \sqrt{x}\frac{\mathrm{d}\sqrt[3]{u}}{\mathrm{d}u}\cdot\frac{\mathrm{d}u}{\mathrm{d}x}\\
&= \frac{1}{2\sqrt{x}}\sqrt[3]{1+x^2} + \sqrt{x}\frac{1}{3u^{\frac{2}{3}}}\cdot\frac{\mathrm{d}(1+x^2)}{\mathrm{d}x}\\
&= \frac{1}{2\sqrt{x}}\sqrt[3]{1+x^2} + \sqrt{x}\frac{1}{3(1+x^2)^{\frac{2}{3}}}\cdot 2x\\
&= \frac{1}{2\sqrt{x}}\sqrt[3]{1+x^2} + \frac{2x^{\frac{3}{2}}}{3(1+x^2)^{\frac{2}{3}}}\\
&= \frac{3+7x^2}{6\sqrt{x}(1+x^2)^{\frac{2}{3}}}.
\end{aligned}$$

(3)

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\ln x\right)^{b^2} &= \frac{\mathrm{d}}{\mathrm{d}x}(u)^{b^2} \qquad\qquad \text{where } u = x^2\ln x\\
&= \frac{\mathrm{d}u^{b^2}}{\mathrm{d}u}\cdot\frac{\mathrm{d}u}{\mathrm{d}x}\\
&= b^2u^{b^2-1}\cdot\frac{\mathrm{d}}{\mathrm{d}x}x^2\ln x\\
&= b^2(x^2\ln x)^{b^2-1}\cdot\left(\frac{\mathrm{d}x^2}{\mathrm{d}x}\ln x + x^2\frac{\mathrm{d}\ln x}{\mathrm{d}x}\right)\\
&= b^2(x^2\ln x)^{b^2-1}\cdot\left(2x\ln x + x^2\frac{1}{x}\right)\\
&= b^2(x^2\ln x)^{b^2-1}\cdot(2x\ln x + x)\,.
\end{aligned}
$$

(4)

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\tan^2\left(\frac{1}{cx^2+d}\right) &= \frac{\mathrm{d}}{\mathrm{d}x}u^2 \qquad\qquad \text{where } u = \tan\left(\tfrac{1}{cx^2+d}\right)\\
&= \frac{\mathrm{d}u^2}{\mathrm{d}u}\cdot\frac{\mathrm{d}u}{\mathrm{d}x}\\
&= 2u\cdot\frac{\mathrm{d}}{\mathrm{d}x}\tan\left(\frac{1}{cx^2+d}\right)\\
&= 2\tan\left(\frac{1}{cx^2+d}\right)\cdot\frac{\mathrm{d}}{\mathrm{d}x}\tan v \qquad\qquad \text{where } v = \tfrac{1}{cx^2+d}\\
&= 2\tan\left(\frac{1}{cx^2+d}\right)\cdot\frac{\mathrm{d}\tan v}{\mathrm{d}v}\cdot\frac{\mathrm{d}v}{\mathrm{d}x}\\
&= 2\tan\left(\frac{1}{cx^2+d}\right)\cdot\sec^2 v\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{cx^2+d}\right)\\
&= 2\tan\left(\frac{1}{cx^2+d}\right)\cdot\sec^2\left(\frac{1}{cx^2+d}\right)\cdot\frac{(cx^2+d)\cdot(1)'-1\cdot(cx^2+d)'}{(cx^2+d)^2}\\
&= \frac{-4cx}{(cx^2+d)^2}\cdot\tan\left(\frac{1}{cx^2+d}\right)\cdot\sec^2\left(\frac{1}{cx^2+d}\right).
\end{aligned}
$$

(5) $\dfrac{d\,(x^2+7)^{(x^4+3)}}{dx} = ?$

logarithmic differentiation

Let $y = (x^2+7)^{(x^4+3)}$. Take log to obtain $\ln y = (x^4+3)\ln(x^2+7)$ ... (*)

Take $\frac{d}{dx}$ of (*) to obtain: $\dfrac{1}{y}\dfrac{dy}{dx} = (x^4+3)'\ln(x^2+7) + (x^4+3)\dfrac{(x^2+7)'}{x^2+7}$

$\therefore \dfrac{dy}{dx} = y\left[(4x^3)\ln(x^2+7) + \dfrac{x^4+3}{x^2+7}(2x)\right] = \cdots$

$b^c = \exp(c\ln b)$

Alternatively $\because (x^2+7)^{(x^4+3)} = \exp\left[(x^4+3)\ln(x^2+7)\right]$

$\therefore \dfrac{d}{dx}\left[(x^2+7)^{(x^4+3)}\right] = \left\{\exp\left[(x^4+3)\ln(x^2+7)\right]\right\}\dfrac{d}{dx}\left[(x^4+3)\ln(x^2+7)\right]$

$= \cdots$

Ex. $4\frac{1}{2}$ Let $x>0$ and $\alpha \in \mathbb{R}$. Then

$$(x^\alpha)' = \alpha\, x^{\alpha-1}.$$

Pf. $\because\ x^\alpha = e^{\alpha \ln x}$,

by the chain rule, we have:

$$(x^\alpha)' = \left.\frac{de^y}{dy}\right|_{y=\alpha \ln x} \frac{dy}{dx} \qquad (\text{where } y=\alpha \ln x)$$

$$= \left. e^y \right|_{y=\alpha\ln x} (\alpha)\frac{1}{x}$$

$$= x^\alpha (\alpha) \frac{1}{x} = \alpha\, x^{\alpha-1}.$$

Thm 6 (L'Hôpital's Rule)

Suppose: f, g are differentiable on (a,b),

$c \in (a,b)$,

$g'(x) \neq 0$ for all $x \in (a,b)\setminus\{c\}$,

$$\lim_{\substack{x\to c^- \\ x\to c^+ \\ x\to c}} f(x) = \begin{matrix} 0 \\ \infty \\ -\infty \end{matrix} = \lim_{\substack{x\to c^- \\ x\to c^+ \\ x\to c}} g(x).$$

$\frac{0}{0}$ form

$\frac{\infty}{\infty}$ form

Then
$$\lim_{\substack{x\to c^- \\ x\to c^+ \\ x\to c}} \frac{f(x)}{g(x)} = \lim_{\substack{x\to c^- \\ x\to c^+ \\ x\to c}} \frac{f'(x)}{g'(x)},$$

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provided the limit on the right $=$ a real number or ∞, or $-\infty$. IV-7

~~III-16~~

Thm 6′ (L'Hôpital's rule)

Suppose: f, g are differentiable on $(-\infty, b)$, (a, ∞)

$g'(x) \neq 0$ for all $x \in (-\infty, b)$, (a, ∞)

$$\lim_{\substack{x\to-\infty \\ x\to\infty}} f(x) = \underset{\substack{\infty \\ -\infty}}{0} = \lim_{\substack{x\to-\infty \\ x\to\infty}} g(x).$$

Then
$$\lim_{\substack{x\to-\infty \\ x\to\infty}} \frac{f(x)}{g(x)} = \lim_{\substack{x\to-\infty \\ x\to\infty}} \frac{f'(x)}{g'(x)},$$

provided the limit on right $=$ a real number, or ∞, or $-\infty$.

We can re-capture the important formulas

Ex. 5. $\lim_{x\to 0} \frac{\sin x}{x} = 1$ and $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e$ by applying Thm 6, Thm 6′.

$$\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$$

$$= \lim_{x\to 0} \frac{\sin(x) - x}{x \sin x} \qquad \left(\text{it's } \tfrac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{\cos(x) - 1}{\sin x + x\cos x} \qquad \left(\text{in } \tfrac{0}{0} \text{ form again}\right)$$

$$= \lim_{x\to 0} \frac{-\sin x}{2\cos(x) - x\sin x} \qquad \left(\text{not in } \tfrac{0}{0} \text{ form}\right)$$

$$= \frac{0}{2} = 0.$$

Ex. 6. Calculate the $\lim_{x\to\infty}\left(\sqrt{x^2+3x}-x\right)$.

Method 1 The required limit

$$=\lim_{x\to\infty}\left(\frac{(x^2+3x)-x^2}{\sqrt{x^2+3x}+x}\right) \qquad \left[\frac{\text{numerator}\cdot\left(\sqrt{x^2+3x}+x\right)}{\text{denominator}\cdot\left(\sqrt{x^2+3x}+x\right)}\right]$$

$$=\lim_{x\to\infty}\frac{3x}{x\left(\sqrt{1+3/x}+1\right)}=\lim_{x\to\infty}\frac{3}{\sqrt{1+3/x}+1}=\frac{3}{2}.$$

Method 2 The required limit

$$=\lim_{x\to\infty}x\left(\sqrt{1+3/x}-1\right)=\lim_{x\to\infty}\frac{\sqrt{1+3/x}-1}{1/x} \quad \left(\text{in } \tfrac{0}{0} \text{ form}\right)$$

$$=\lim_{x\to\infty}\frac{\frac{1}{2}(1+3/x)^{-1/2}(-3/x^2)}{-1/x^2} \quad (\text{by L'Hospital's rule})$$

$$=\lim_{x\to\infty}\frac{3}{2}(1+3/x)^{-1/2}=\frac{3}{2}.$$

V-6(+1)

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Ex. 1 Find

$$\lim_{x\to 0} (\cos x)^{1/x^2} \quad (= e^{-1/2}).$$

Soln

Let $y = x^{-2} \ln(\cos x)$. Then

$$\lim_{x\to 0} y = \lim_{x\to 0} \frac{\ln(\cos x)}{x^2} \qquad (\text{in } \tfrac{0}{0} \text{ form})$$

$$= \lim_{x\to 0} \frac{\frac{1}{\cos x}(-\sin x)}{2x}$$

$$= \lim_{x\to 0} \frac{-\sin x}{2x\cos x} \qquad (\text{in } \tfrac{0}{0} \text{ form})$$

$$= \lim_{x\to 0} \frac{-\cos x}{2\cos x - 2x\sin x} \qquad (\text{not in } \tfrac{0}{0} \text{ form})$$

$$\lim_{x\to 0} y = -\tfrac{1}{2}.$$

Because $(\cos x)^{1/x^2} = e^y$ and e^y is continuous on $\mathbb{R}$,

$$\lim_{x\to 0} (\cos x)^{1/x^2} = \lim_{y\to -\frac{1}{2}} e^y = e^{-1/2}.$$

Ex. 2 Suppose $p>0$. Then

$$\lim_{x\to\infty} \frac{x^p}{e^x} = \lim_{x\to\infty} \frac{p\,x^{p-1}}{e^x} = \begin{cases} 0 & \text{if } p\le 1 \\ \text{proceed on} & \text{if } p>1 \end{cases}$$

(after finitely many steps.) $= 0$.

Ex. 3. Suppose $p>0$. Then

$$\lim_{x\to\infty} \frac{\ln x}{x^p} = \lim_{x\to\infty} p^{-1} x^{-p} = 0.$$

Implicit differentiation

Ex. 1 We can regard y as an implicit function of x by the equation:

$$x^2 - 2xy + 3y^2 = 5.$$

Taking differentiation of x, we get

$$2x - 2y - 2x\frac{dy}{dx} + 6y\frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = \frac{x-y}{x-3y}.$$

Suppose we are given that $y = \frac{1+\sqrt{13}}{3}$ at $x=1$

Then

$$\left.\frac{dy}{dx}\right|_{x=1} = \left.\frac{x-y}{x-3y}\right|_{\substack{x=1\\ y=\frac{1+\sqrt{13}}{3}}} = \frac{1}{3} - \frac{2\sqrt{13}}{39}.$$

This agrees with (explicit) differentiation of the function

$$y = \frac{x+\sqrt{15-2x^2}}{3} \quad \left(y = \frac{1+\sqrt{13}}{3} \text{ at } x=1\right).$$

Differentials

$dx \ (= \Delta x)$

$dy = y'dx \ (= y'\Delta x)$ [thus, formally, $dy/dx = y'$]

$\Delta y = y(x+\Delta x) - y(x)$ (= actual change in y)

dy is an approximation of Δy.

rules d (a constant) $= 0$

$d(y+z) = dy + dz$

$d(yz) = (dy)z + y\,dz$ [by the product rule of derivatives:
$(yz)' = y'z + yz'$, so $(yz)'dx = (y'dx)z + y(z'dx)$.]

We can apply differentials in doing implicit differentiation as the following example shows.

Ex. Find $\frac{dy}{dx}$ where $x^2 - 2xy + 3y^2 = 5$. $(*)$

From $(*)$ (by applying d): $d(x^2) - d(2xy) + d(3y^2) = d5$.

By the preceding rules,

$$2x\,dx - 2(dx)y - 2(dy)x + 6y\,dy = 0$$

i.e.

$$2(x-y)\,dx = 2(x-3y)\,dy$$

$$\therefore \quad \frac{dy}{dx} = \frac{x-y}{x-3y},$$

which confirms our result in Ex. 1 above (of the section under implicit differentiation).