§ Definite Integrals

Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$.

A partition P of $[a,b]$ is a collection of points $\{x_0, x_1, \cdots, x_{j-1}, x_j, \cdots, x_n\} \subset [a,b]$ such that

$$x_0 = a < x_1 < \cdots < x_{j-1} < x_j < \cdots x_n = b.$$

The norm of the partition P ($\|P\|$) is defined to be

$$\max\{x_j - x_{j-1} : j = 1, 2, \cdots, n\}$$

i.e. the maximum length of the (sub-division) intervals $[x_{j-1}, x_j]$, $j = 1, 2, \cdots, n$.

The right-sum is defined to be

$$\begin{aligned} R(f;P) &= f(x_1)(x_1 - x_0) + f(x_2)(x_2 - x_1) + \cdots + f(x_n)(x_n - x_{n-1}) \\ &= f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + \cdots + f(x_n)\Delta x_n \\ &= \sum_{j=1}^{n} f(x_j)\Delta x_j , \end{aligned}$$

where $\Delta x_j = x_j - x_{j-1}$.

The <u>left-sum</u> is

$$L(f;P) = \sum_{j=1}^{n} f(x_{j-1})\,\Delta x_j\,;$$

the <u>mid-point sum</u> is

$$M(f;P) = \sum_{j=1}^{n} f\left(\frac{x_{j-1}+x_j}{2}\right) \Delta x_j.$$

It is proved that there exists a number I such that each of $R(f;P)$, $L(f;P)$, $M(f;P)$ is very close to I whenever the norm $\|P\|$ of the partition P (of $[a,b]$) is small enough, i.e. for each $\varepsilon>0$, there exists $\delta>0$ such that

$$|R(f;P)-I|<\varepsilon,\quad |L(f;P)-I|<\varepsilon,\quad |M(f;P)-I|<\varepsilon$$

whenever $\|P\|<\delta$.

This number I is denoted by

$$\int_a^b f(x)\,dx\,, \quad \text{or} \quad \int_a^b f\,.$$

and called the (definite) integral of f over $[a,b]$.

If $f(x) \geq 0$ for all $x \in [a,b]$, then we can interpret $\int_a^b f(x)\,dx$ as the area of the region underneath ~~the graph of f~~ over $[a,b]$.
bounded by $y=0$, $y=f(x)$, $x=a$ and $x=b$

If $f(x) \leq 0$ for all $x \in [a,b]$, then $\int_a^b f(x)\,dx$ may be interpreted as the signed (or the negative of the) area of the region bounded by $y=0$, the graph of f, $x=a$ and $x=b$.

Basic Properties

Suppose f is continuous on the relevant intervals.

1) $\int_a^a f(x)\,dx = 0$.

2) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$. (a really definition)

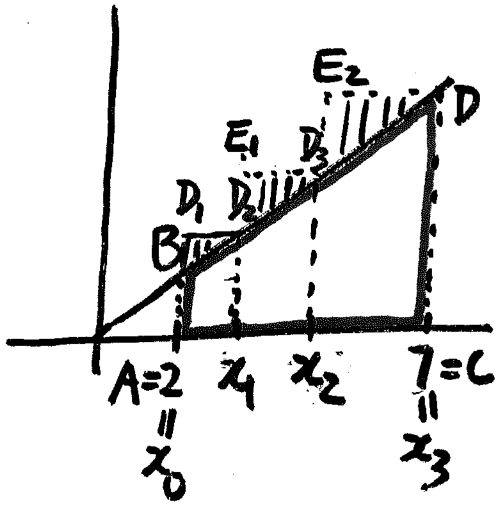
3) $\int_a^b k f(x)\,dx = k \int_a^b f(x)\,dx$, where k is a constant.

4) $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$.

5. $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

Example

Let us consider $f(x)=x$, $a=2$, $b=7$.



Let P: $x_0=2<x_1<x_2<x_3=7$

$$\Big|R(f;P) - \underbrace{\text{area of } \square ABDC}_{\frac{1}{2}(2+7)(7-2)}\Big| \quad \begin{matrix}\text{trapezoid}\\ \text{(trapezium)}\end{matrix}$$

$$\doteqdot \text{area}(\triangleright BD_1D_2) + \text{area}(\triangleright D_2E_1D_3) + \text{area}(\triangleright D_3E_2D)$$

$$\doteqdot \frac{1}{2}BD_1\cdot D_1D_2 + \frac{1}{2}D_2E_1\cdot E_1D_3 + \frac{1}{2}D_3E_2\cdot E_2D$$

The max. of $\{BD_1, D_2E_1, D_3E_2\}$
$= D_3E_2$

$$\le \frac{1}{2}D_3E_2\cdot(D_1D_2+E_1D_3+E_2D)$$

$$= \frac{1}{2}D_3E_2\cdot(7-2) = \frac{5}{2}\underbrace{D_3E_2}_{\|P\|}$$

Note that if $\|P\|$ is small enough (so a lot of division points, and each Δx_j is small), then the corresponding D_3E_2 (which is the maximum of $|f(x_j)-f(x_{j-1})|$) is very small.

$$\therefore \int_2^7 x\,dx = \frac{1}{2}(2+7)(7-2) = \frac{1}{2}(7^2-2^2)$$

$$= \frac{1}{2}x^2\Big|_{x=2}^{x=7} \qquad \begin{matrix}\| \\ \frac{45}{2}\end{matrix} .$$

NOT required (taken from Davidson & Donsig, Real Analysis w. Real Applications)

EXAMPLE. Consider the function $f(x) = x^p$ on $[a, b]$, where $p \neq -1$ and $0 < a < b$. Take the partition $P_n = \{a = x_0 < x_1 < \dots < x_n = b\}$, where $x_j = a\left(\frac{b}{a}\right)^{j/n}$ for $0 \leq j \leq n$.

To keep the notation under control, let $R = (b/a)^{1/n}$. For example, $x_j = aR^j$ and $\Delta_j = x_j - x_{j-1} = aR^{j-1}(R-1)$.

So for $p > 0$, we have $R(f, P_n) = R^p L(f, P_n)$ and

$$\begin{aligned} L(f, P_n) &= \sum_{j=1}^{n} f(x_{j-1})\,(x_j - x_{j-1}) \\ &= \sum_{j=1}^{n} a^p R^{p(j-1)} a R^{j-1}(R-1) \\ &= a^{p+1}(R-1)\sum_{j=0}^{n-1} R^{(p+1)j}. \end{aligned}$$

Summing the geometric series and rearranging, we have

$$\begin{aligned} L(f, P_n) &= a^{p+1}(R-1)\frac{R^{n(p+1)}-1}{R^{p+1}-1} \\ &= a^{p+1}(R-1)\frac{\left(\frac{b}{a}\right)^{p+1}-1}{R^{p+1}-1} \\ &= (b^{p+1}-a^{p+1})\frac{R-1}{R^{p+1}-1}. \end{aligned}$$

We will take a limit as $n \to +\infty$. To show the role of n clearly, we set $r = \frac{b}{a}$ and $h = 1/n$, so that $R = r^{1/n} = r^h$. The key is to recognize the limit as the computation of two derivatives.

$$\begin{aligned} \lim_{n\to\infty} L(f, P_n) &= (b^{p+1}-a^{p+1}) \lim_{n\to\infty} \frac{r^{1/n}-1}{r^{(p+1)/n}-1} \\ &= (b^{p+1}-a^{p+1}) \lim_{h\to 0} \frac{r^h - 1}{h}\,\frac{h}{r^{(p+1)h}-1} \\ &= (b^{p+1}-a^{p+1}) \frac{\frac{d}{dx}(r^x)|_{x=0}}{\frac{d}{dx}(r^{(p+1)x})|_{x=0}} \\ &= (b^{p+1}-a^{p+1}) \frac{\log r}{(p+1)\log r} = \frac{b^{p+1}-a^{p+1}}{p+1}. \end{aligned}$$

Since $R(f, P_n) = \left(\frac{a}{b}\right)^{p/n} L(f, P_n)$ has the same limit, we conclude that

$$\frac{b^{p+1}-a^{p+1}}{p+1} \leq L(f) \leq U(f) \leq \frac{b^{p+1}-a^{p+1}}{p+1}.$$

So this function is Riemann integrable with $\displaystyle\int_a^b x^p\,dx = \frac{b^{p+1}-a^{p+1}}{p+1}$.

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Fundamental Theorem of Calculus

If $F(x)$ is an anti-derivative of $f(x)$, i.e.

$$F'(x) = f(x) \quad \text{on } [a,b],$$

and if $\underline{f(x) \text{ is continuous on } [a,b]}$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

<u>Example</u> $f(x) = x$, $a=2$, $b=7$.

Let $F(x) = \frac{1}{2}x^2$. Then $F'(x) = x = f(x)$ on $[2,7]$.

And we have $\int_2^7 x\,dx = \frac{1}{2}(7^2 - 2^2) = F(7) - F(2)$.

In fact, for any $a<b$ in $\mathbb{R}$,

$$\int_a^b x\,dx = F(b) - F(a).$$

<u>Heuristic argument for the theorem</u>

$$
\begin{aligned}
& f(c_j)\Delta x_j && \text{(where } c_j \text{ is a point in } [x_{j-1}, x_j]) \\
&= F'(c_j)\Delta x_j && \text{(by assumption, } F'(x) = f(x)) \\
&\approx F(x_j) - F(x_{j-1}) && \text{(by Mean Value Thm)}
\end{aligned}
$$

$$\therefore \sum_{j=1}^{n} f(c_j)\Delta x_j \approx \sum_{j=1}^{n} [F(x_j) - F(x_{j-1})] = F(x_n) - F(x_0) = F(b) - F(a)$$

More Examples

1. $\displaystyle\int_0^3 x^2\,dx = \frac{1}{3}x^3\Big|_{x=3} - \frac{1}{3}x^3\Big|_{x=0} = 9.$

2. $\displaystyle\int_1^3 \left(2x + 3e^x - \frac{4}{x}\right) dx$

$$= 2\int_1^3 x\,dx + 3\int_1^3 e^x\,dx - 4\int_1^3 \frac{1}{x}\,dx$$

$$= 2\cdot\frac{1}{2}x^2\Big|_1^3 + 3\cdot e^x\Big|_1^3 - 4\ln x\Big|_1^3$$

$$= 8 + 3e\,(e^2-1) - 4\ln 3.$$

3. $\displaystyle\int_{-4}^1 \sqrt{5-t}\,dt$

$$= \int \sqrt{5-t}\,dt\,\Big|_{-4}^{1} = \int u^{1/2}\frac{dt}{du}\,du\,\Big|_{t=-4}^{t=1} \quad \left(\begin{array}{l}\text{with } u = 5-t \\ \frac{du}{dt} = -1\end{array}\right)$$

$$= -\int u^{1/2}\,du\,\Big|_{t=-4}^{t=1} = \frac{-2}{3}\left(u^{3/2}+c\right)\Big|_{-4}^{t=1} = \frac{-2}{3}(5-t)^{3/2}\Big|_{-4}^{t=1}$$

$$= -\frac{2}{3}\left(4^{3/2} - 9^{3/2}\right) = -\frac{2}{3}(8-27) = \frac{38}{3}.$$

alternatively

$$= -\int_9^4 u^{1/2}\,du = -\frac{2}{3}\left(4^{3/2} - 9^{3/2}\right) = \frac{38}{3}.$$

$u = \begin{cases} 4 & \text{when } t = 1 \\ 9 & \text{when } t = -4. \end{cases}$

Average

Given 10 numbers $a_1, a_2, \cdots, a_{10}$,

their average $= \frac{1}{10}(a_1 + a_2 + \cdots + a_{10})$.

Given a continuous function f on $[a,b]$, how should its average over $[a,b]$ be defined?

Consider a partition

$$a = x_0 < x_1 < \cdots < x_n = b$$

with $\Delta x = x_1 - x_0 = x_2 - x_1 = \cdots = x_n - x_{n-1}$.

$$\frac{1}{n} \sum_{k=1}^{n} f(x_k) = \frac{1}{n(\Delta x)} \sum_{k=1}^{n} f(x_k)\,\Delta x$$

$$= \frac{1}{b-a} \sum_{k=1}^{n} f(x_k)\Delta x$$

$$\rightsquigarrow \frac{1}{b-a} \int_a^b f(x)\,dx \quad \text{as } n \to \infty$$

So, we may consider/define

$\frac{1}{b-a}\int_a^b f(x)\,dx$ as the average of f over $[a,b]$.

2nd mean value thm for integrals.

Thm Suppose f is continuous on $[a, b]$,
g is continuous on $[a, b]$, and
of one sign on $[a, b]$ (+ throughout or − throughout).

Then there exists $\xi \in [a, b]$ such that

$$\int_a^b f(x)\, g(x)\, dx = f(\xi) \int_a^b g(x)\, dx .$$

(first) Mean Value thm for integrals

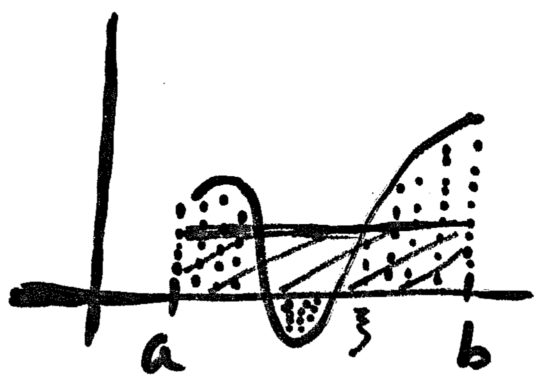
Corollary Suppose that f is continuous on $[a, b]$.

Then there exists $\xi \in [a, b]$ such that

$$\int_a^b f(x)\, dx = f(\xi)\,(b-a),$$

i.e. $$\frac{1}{b-a} \int_a^b f(x)\, dx = f(\xi)$$

($a < b$ understood).



e.g.,

$$\frac{1}{2-1}\int_1^2 x^2\,dx \quad \left(=\frac{1}{3}(2^3-1^3)=\frac{7}{3}\right)$$

$$=\left(\sqrt{\frac{7}{3}}\right)^2,$$

$$\frac{1}{2}\int_0^2 x^3\,dx \quad \left(=\frac{1}{8}\cdot 2^4=2\right)$$

$$=\left(2^{\frac{1}{3}}\right)^3.$$

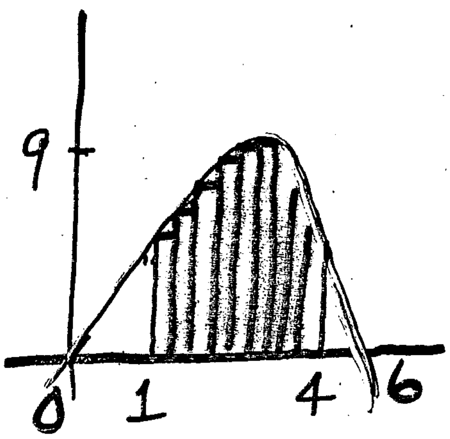
Applications

Area between curves

Examples

$f(x)=6x-x^2$

1. Find the area bounded by $f(x)=6x-x^2$ and $y=0$, for $x\in[1,4]$.



The required area

$$= \int_1^4 f(x)\,dx$$

$$= \int_1^4 (6x-x^2)\,dx$$

$$= 6\left(\frac{x^2}{2}\Big|_1^4\right) - \left(\frac{1}{3}x^3\Big|_1^4\right)$$

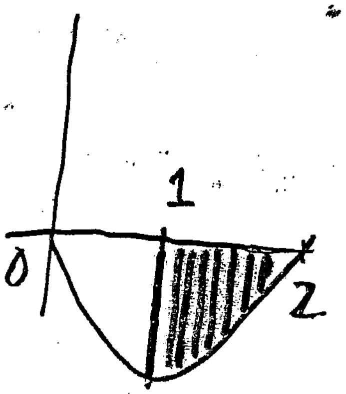
$$= \cdots = 24.$$

2. Find the area bounded by

$$f(x) = x^2 - 2x$$

over (A) $[1, 2]$, (B) $[-1, 1]$.

[A]

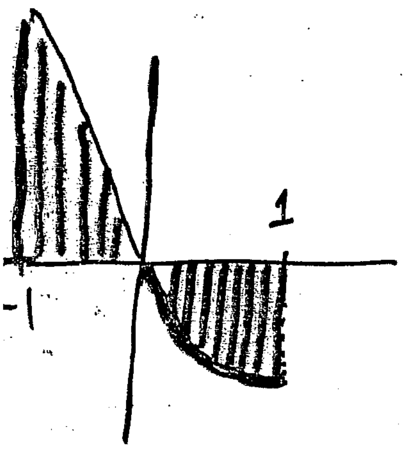


The required area

$$= -\int_1^2 f(x)\,dx \quad \left(\text{as } f(x) \le 0 \text{ for } x \in [1,2]\right)$$

$$= \frac{2}{3}$$

[B]



The required area

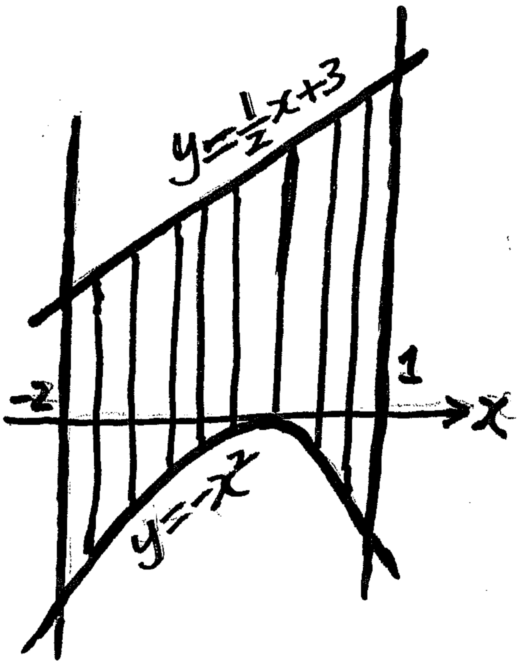
$$= \cancel{\int_{-1}^{1} f(x)\,dx}$$

$$= \int_{-1}^{0} f(x)\,dx + \int_0^1 [-f(x)]\,dx$$

3. Find the area between / bounded by the graphs of

$$f(x) = \frac{1}{2}x + 3, \quad g(x) = -x^2, \quad x = -2$$

and $x = 1$.



The required area

$$= \int_{-2}^{1} [f(x) - g(x)]\,dx$$

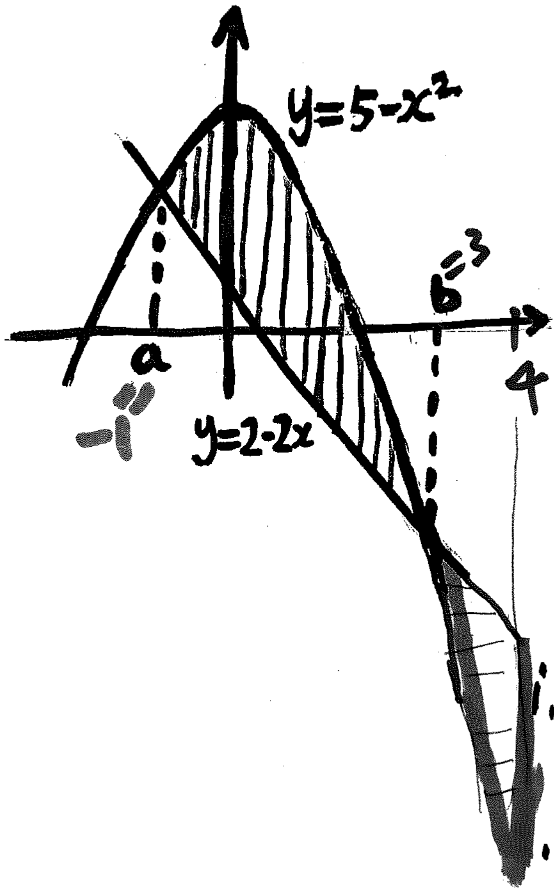
(as $g(x) \le f(x)$ on $[-2, 1]$)

$$= \int_{-2}^{1} \left(+x^2 + \frac{x}{2} + 3\right) dx$$

$$= \cdots = \frac{33}{4}.$$

4. Find the area between enclosed by the graphs of

$f(x) = 5 - x^2$ and $g(x) = 2 - 2x$.



By the graph, the required area is given by

$$\int_a^b [f(x) - g(x)]\,dx$$

where a, b are solutions of

$g(x) = f(x)$,

i.e. $x^2 - 2x - 3 = 0$

$\therefore\ a = -1,\ b = 3$

$$\therefore \int_a^b [f(x) - g(x)]\,dx$$

$$= \int_{-1}^{3} (-x^2 + 2x + 3)\,dx$$

$$= \cdots = \frac{32}{3}.$$

5. Find the area bounded by the graphs of

$f(x) = 5 - x^2$, $g(x) = 2 - 2x$, and $x = 4$.

(cf. that of the previous example)

By the graph, the required area is

$$\int_{-1}^{3} [f(x) - g(x)]\,dx + \int_{3}^{4} \underline{[g(x) - f(x)]}\,dx \quad (\text{why?})$$

$$= \frac{32}{3} + \int_{3}^{4} (x^2 - 2x - 3)\,dx$$

$$= \frac{32}{3} + \frac{7}{3} = 13.$$

<u>Thm</u> Suppose f is continuous on $[a,b]$.
Then the function $I : [a,b] \to \mathbb{R}$ given by

$$I(x) = \int_a^x f(t)\,dt \ , \quad x \in [a,b]$$

is differentiable on $[a,b]$, and

$$I'(x) = f(x), \quad x \in [a,b]$$

i.e. $$\boxed{\frac{d}{dx} \int_a^x f(t)\,dt = f(x), \quad x \in [a,b]}$$

<u>Proof</u> Let $x \in [a,b)$, $h \in (0, b-x)$. Then

$$I(x+h) - I(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt$$

$$= \int_x^{x+h} f(t)\,dt = h\, f(\xi) \quad \text{for some } \xi \in [x, x+h].$$

(Explanation is given for the last equality in class)

Hence $$\lim_{h \to 0^+} \frac{I(x+h) - I(x)}{h} = \lim_{h \to 0^+} f(\xi) = f(x),$$

i.e. $I'_+(x) = f(x)$ for all $x \in [a,b)$.

Similarly, $I'_-(x) = f(x)$ for all $x \in (a,b]$.

$\therefore \quad I'(x) = f(x)$ for all $x \in [a,b]$.

Thm Suppose f is continuous on $[a,b]$. Then

$$\boxed{\frac{d}{dx}\int_x^b f(t)\,dt = -f(x), \quad x \in [a,b].}$$

Examples

1. $$\int_a^x \sin t\,dt = -\cos t\Big|_{t=a}^{x} = \cos a - \cos x$$

and $$\frac{d}{dx}\int_a^x \sin t\,dt = \frac{d}{dx}(\cos a - \cos x) = \sin x.$$

Similarly, $$\int_x^b \sin t\,dt = -\cos b + \cos x,$$

and $$\frac{d}{dx}\int_x^b \sin t\,dt = -\sin x.$$

2. $$\frac{d}{dx}\int_x^2 e^{-t^2}\,dt = -e^{-x^2},$$

even though it is impossible to express $\int e^{-t^2}dt$ in terms of (elementary) functions we have met so far (by means arithmetic operations & rational powers)

Arc-lengths, Volumns, and Surface Areas

A non-self-intersecting arc in the plane is, by definition, a function $F: [a,b] \to \mathbb{R}^2$ such that for all $x_1 \neq x_2$ in $[a,b]$, $F(x_1) \neq F(x_2)$ [except perhaps when $\{x_1, x_2\} = \{a, b\}$].

There is good mathematical sense to define an arc as a function F, <u>**NOT**</u> the range $\{F(x) : x \in [a,b]\}$.
NOT the graph $\{(x, F(x)) : x \in [a,b]\}$.

<u>Example 1</u> The upper semi-circle of radius r is given by:

$$f(x) = \sqrt{r^2 - x^2}, \quad x \in [-r, r].$$

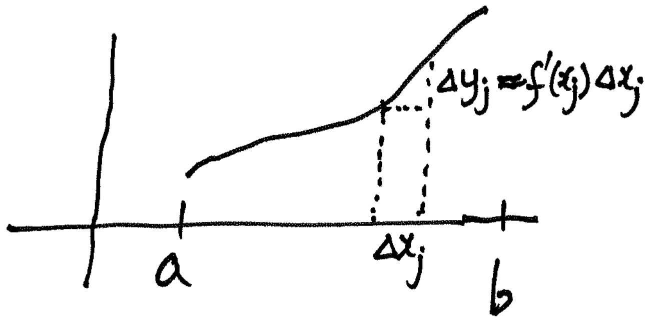
By this we mean the function $F: [-r, r] \to \mathbb{R}^2 : x \mapsto (x, f(x))$.

Thus, for each function $f: [a,b] \to \mathbb{R}$, we can consider the function $F: [a,b] \to \mathbb{R}^2 : x \mapsto (x, f(x))$. Then F is a non-self-intersecting arc in the above sense. By abuse of language, we refer f as an arc [instead of the $F: x \mapsto (x, f(x))$]. Often, we assume f has continuous derivative f' on $[a,b]$.

The <u>length of such an arc</u> is given by:

$$\int_a^b \sqrt{1 + (f'(x))^2} \, dx.$$

The reason for such a definition can be illustrated by the following diagram:



<u>Example 2</u> For the arc given in Example 1 above, the arc-length is computed as follows:

$$\int_{-r}^{r} \sqrt{1+\left(\frac{d}{dx}\sqrt{r^2-x^2}\right)^2}\, dx$$

$$= \int_{-r}^{r} \sqrt{1+\left[\frac{1}{2}(r^2-x^2)^{-1/2}(-2x)\right]^2}\, dx$$

$$= \int_{-r}^{r} \sqrt{1+x^2(r^2-x^2)^{-1}}\, dx$$

$$= \int r\, d\theta \Big|_{x=-r}^{x=r}$$

[by substitution $x = r\cos\theta$, $\theta \in [-\pi, 0]$: then $\sqrt{1+x^2(r^2-x^2)^{-1}} = \frac{1}{-\sin\theta}$, $\frac{dx}{d\theta} = -r\sin\theta$]

$$= r \arccos\frac{x}{r}\Big|_{x=-r}^{x=r}$$

$$= r\left[\arccos(1) - \arccos(-1)\right]$$

$$= r\left[0-(-\pi)\right] = \pi r.$$

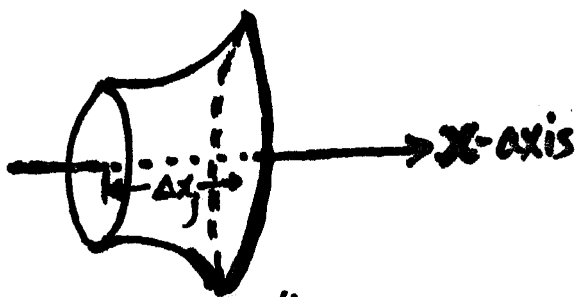
Therefore, the length of a circle with radius r

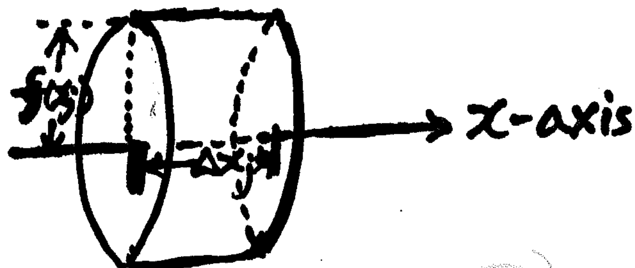
$= 2\pi r.$

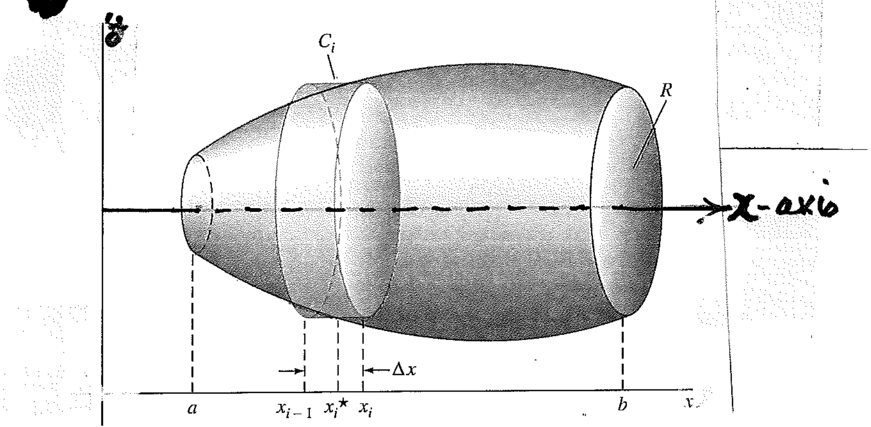
Suppose $f(x) \geqslant 0$ for all $x \in [a, b]$, and suppose we revolve, about the x-axis, the region bounded by the graph of f, the lines $x=a$, $x=b$ and $y=0$. Then the region sweeps out a solid, called a solid of revolution. The volume of this solid is given by:

$$\int_a^b \pi\,(f(x))^2\,dx.$$

The reason for this definition can be illustrated by the following diagram:







VII 3

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Example 3 The volume of the solid obtained by revolving, about the x-axis, the region bounded by the arc given in Example 1 above, and the lines $x=-r$, $x=r$ and $y=0$, is computed as follows:

$$\int_{-r}^{r} \pi (f(x))^2 \, dx$$

$$= \pi \int_{-r}^{r} (r^2 - x^2) \, dx$$

$$= \pi \left[r^2(2r) - \frac{1}{3}(r^3 - (-r)^3) \right]$$

$$= \frac{4}{3} \pi r^3 .$$

Example 4 The volume of the solid obtained by revolving, about the y-axis, the region described in Example 3 above, is computed as follows:

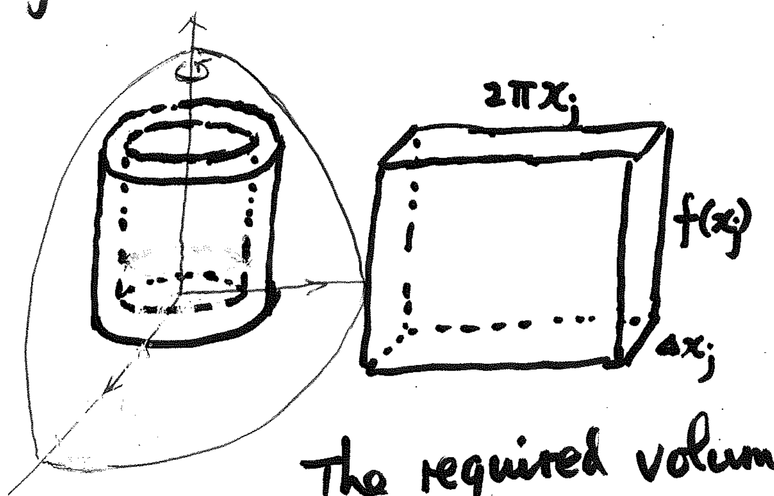
$$\int_0^r \pi\left[f^{-1}(y)\right]^2 dy$$

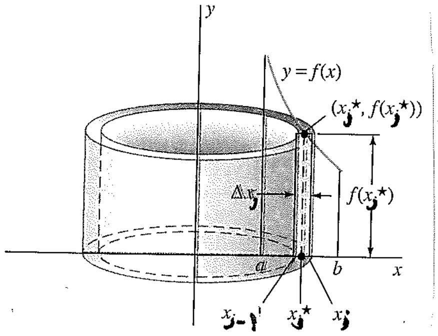
$$= \int_0^r \pi(r^2 - y^2)\,dy$$

$$= \pi\left[r^3 - \frac{1}{3}(r^3)\right] = \frac{2}{3}\pi r^3 .$$

The shell method

Example 5 Find the same volume of Example 4 by the shell method.





$$\text{The required volume} = \int_0^r 2\pi x f(x)\,dx$$

$$= -\pi \int u^{1/2}\,du \,\Big|_{u=r^2}^{0} \qquad (\text{where } u = r^2 - x^2)$$

$$= \frac{2}{3}\pi r^3 .$$

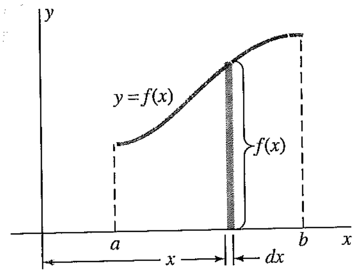
Example 6 Find the volume V of the solid generated by revolving the region under $y = 3x^2 - x^3$, from $x=0$ to $x=3$, around the y-axis.

We draw the picture (which looks like the plots below).

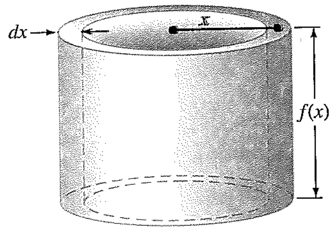
The required volume $= \int_0^3 2\pi x\, y\, dx$

$= \int_0^3 2\pi (3x^3 - x^4)\, dx = \cdots$

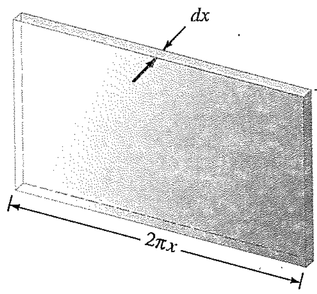
$= 24.3\pi$.



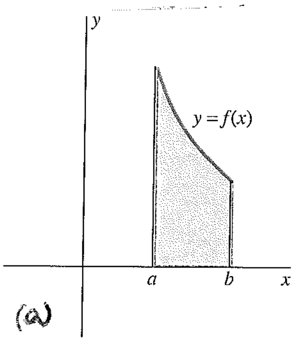
Heuristic device for setting up the equation (*)



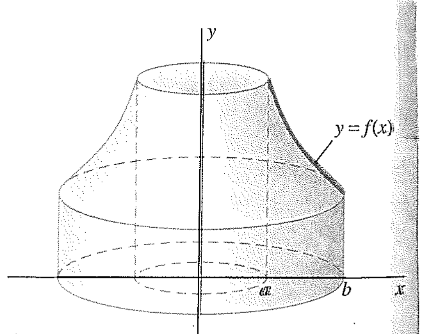
Cylindrical shell of very small thickness.



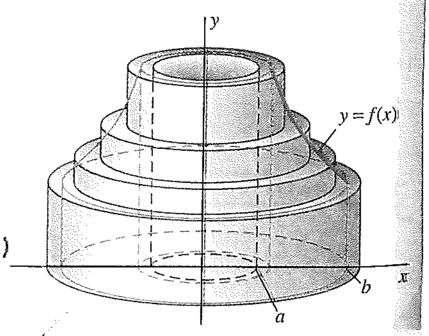
very thin cylindrical shell, flattened out.



(a)



(b)



(c)

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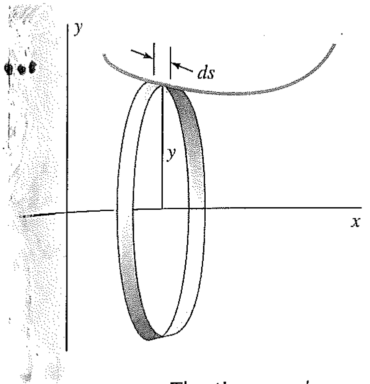
Example 6 Find the volume V of the solid generated by revolving ~~around~~ the region under $y=3x^2-x^3$, from $x=0$ to $x=3$, around the y-axis.

Draw the picture.

The required volume $= \displaystyle\int_0^3 2\pi x y\,dx$

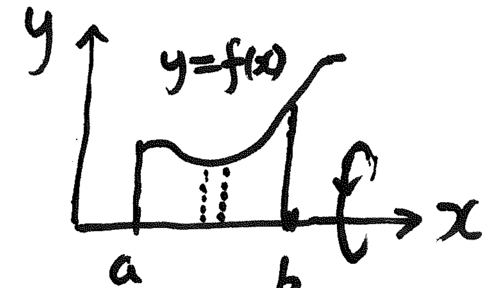
$$= \int_0^3 2\pi(3x^3-x^4)\,dx = \cdots$$

$$= 24.3\,\pi .$$



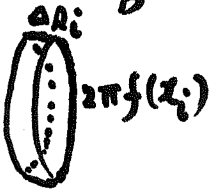
The tiny arc ds generates a ribbon with circumference $2\pi y$ when it is revolved around the x-axis.

Areas of Surfaces of revolution



$$\sum_j 2\pi f(x_j)\,\Delta A_j \;\rightarrow\; \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$$

$$\Delta A_j = \sqrt{1+[f'(x_j)]^2}\;\Delta x_j$$



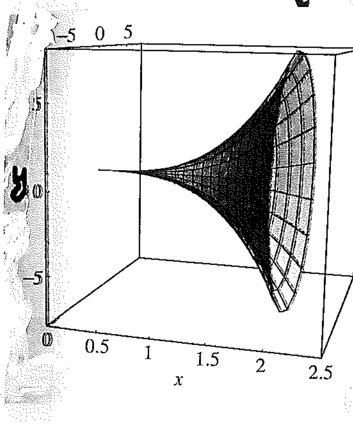
Example 7 Find the area of the surface generated by revolving the curve $y=x^3$, $x\in[0,2]$ around the x-axis.

The required area $= \displaystyle\int_0^2 2\pi x^3\sqrt{1+[(x^3)']^2}\,dx$

$$= \int_0^2 2\pi x^3\sqrt{1+9x^4}\,dx$$

$$= \cdots$$

$$= \frac{\pi}{27}\left[(145)^{3/2}-1\right] \simeq 203 .$$



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