§ Indefinite Integrals — Antiderivatives

$(x^2)' = 2x$

$(\frac{1}{2}x^2)' = x$

$(\frac{1}{2}x^2 + 3)' = x$

$(\frac{1}{2}x^2 + c)' = x$

an arbitrary constant

$\int x\,dx = \frac{1}{2}x^2 + c$

Mean value theorem implies the following.

Let $f, g, h : (a,b) \to \mathbb{R}$.

If $\underset{\text{on }(a,b)}{g' = f}$ then " $\underset{\text{on }(a,b)}{h' = f}$ iff $\overset{\text{on }(a,b)}{h = g + c}$ "

a constant

[Equivantly: Let $k : (a,b) \to \mathbb{R}$.

If $k'(x) = 0$ for all $x \in (a,b)$,

then $k(x)$ is a constant function on (a,b).]

The expression

" $\underset{\text{on }(a,b)}{\int f(x)\,dx = g(x) + c}$, where c stands for any arbitrary constant "

means that " $g'(x) = f(x)$ for all $x \in (a,b)$ ".

So,

$$\int x^3 dx = \frac{1}{4}x^4 + c \;; \qquad \int dx = \int 1\, dx = x + c \;;$$

$$\rightarrow \int \cos x\, dx = \sin x + c \;; \qquad \int \sin x\, dx = -\cos x + c \;;$$

$$\rightarrow \int e^x dx = e^x + c \;; \qquad \int \frac{1}{x} dx = \ln|x| + c, \text{ for } x \neq 0;$$

$$\rightarrow \int x^p dx = \frac{1}{p+1} x^{p+1} + c \quad \begin{cases} \text{for } x > 0, & \text{if } p \text{ is not an integer} \\ \text{for } x \neq 0, & \text{if } p \neq -1 \text{ is an integer} \end{cases}$$

[In particular, $\int dx = x + c$.]

Terminology

$\int f$, or $\int f(x)\,dx$, is called the <u>indefinite integral</u> of f

Any g such that $g' = f$ is called an <u>anti-derivative of f</u>

We have $\boxed{\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x), \quad \int g'(x)\,dx = g(x) + c}$ (c an arbitrary constant)

$[\int f]' = f$, $\int (g') = g + c$

Any rule of differentiation gives rise to a rule on indefinite integrals e.g.

$$(g+h)' = g' + h' \rightsquigarrow \boxed{\int (f(x)+k(x))\,dx = \int f(x)\,dx + \int k(x)\,dx}$$

$$(cg)' = c(g') \text{ if } c \in \mathbb{R} \rightsquigarrow \boxed{\int c f(x)\,dx = c \int f(x)\,dx}$$

$$(gh)' = g'h + gh' \rightsquigarrow gh = \int g'h + \int gh',$$

or $\boxed{\int g'h = gh - \int gh'}$

Integration by parts

$u = u(x)$

$$\frac{d}{dx} f(u) = f'(u) \frac{du}{dx}$$

$$\int f'(u) \frac{du}{dx}\,dx = f(u) + c \quad \begin{pmatrix}\text{arb.}\\ \text{const.}\end{pmatrix}$$

$\int f'(u)\,du$

$f(u(x))$

$$\boxed{\int \frac{df(u)}{du} \frac{du}{dx}\,dx = f(u) + c \underset{(\dagger)}{=} \int \frac{df(u)}{du}\,du = f(u(x)) + c}$$

Integration by substitution

or .. .. change of variables

Examples

1. $\int (4x^3+2x-1)\,dx$

$= \int 4x^3\,dx + \int 2x\,dx - \int 1\,dx$

$= x^4 + x^2 - x + \underline{\underline{c}}$.

2. $\int 5\sqrt[3]{u^2}\,du = 5\int u^{2/3}\,du = 5\left(\frac{1}{2/3+1}\right)u^{2/3+1} + c$

$= 3\,u^{5/3} + c$.

3. $\int \underline{x^2}\, e^{x^3-1}\,dx$

$= \int \frac{1}{3}\, \underbrace{e^u}_{\frac{de^u}{du}}\, \frac{du}{dx}\,dx$ (where $u = x^3-1$)

$\frac{de^u}{dx}$

(Integration by change of variables)

$= \frac{1}{3}e^u + c = \frac{1}{3}e^{x^3-1} + c$.

4. $\int \underset{h}{x}\, \underset{g'}{e^x}\,dx = x e^x - \int \left(\frac{dx}{dx}\right) e^x\,dx$

$h\,g - \int h' g$ (Integration by part)

with $g = e^x$

$= x e^x - \underline{e^x + c}$ $\int e^x dx$

5. $\int (x^2+2x+5)^5 (2x+2)\,dx$

$= \int u^5 \frac{du}{dx}\,dx$ (where $u = x^2+2x+5$)

By substitution

$= \frac{1}{6}u^6 + c = \frac{1}{6}(x^2+2x+5)^6 + c$.

6. $\int \frac{1}{4x+7}\,dx = \frac{1}{4}\int \frac{1}{u}\frac{du}{dx}\,dx$ (where $u = 4x+7$)

By substitution

$= \frac{1}{4}\ln|u| + c = \frac{1}{4}\ln|4x+7| + c$

7. $\int 4x^2\sqrt{x^3+5}\,dx$

By substitution

$= \frac{4}{3}\int \sqrt{u}\frac{du}{dx}\,dx$ (where $u = x^3+5$)

$= \frac{4}{3}\int u^{1/2}\,du = \frac{4}{3}\cdot\frac{2}{3}u^{3/2} + c = \frac{8}{9}(x^3+5)^{3/2} + c$.

8. $\int \frac{x}{\sqrt{x+2}}\,dx$

$= \int \frac{u-2}{\sqrt{u}}\frac{du}{dx}\,dx$ (where $u = x+2$) By substitution

$= \int (u^{1/2} - 2u^{-1/2})\,du = \int u^{1/2}\,du - 2\int u^{-1/2}\,du$

$= \frac{2}{3}u^{3/2} - 4u^{1/2} + c = \frac{2}{3}(x+2)^{3/2} - 4(x+2)^{1/2} + c$

Further Examples

1. $\displaystyle\int \frac{dx}{a^2+x^2}$ (where $a \neq 0$)

$$= \frac{1}{a^2}\int \frac{1}{1+\left(\frac{x}{a}\right)^2}\,dx \qquad \left(\text{with } t=\frac{x}{a},\ \frac{dx}{dt}=a,\ dx = a\,dt\right)$$

read (4) on (VI.3) from right to left

$$= \frac{1}{a^2}\int \frac{1}{1+t^2}\,\frac{dx}{dt}\,dt = \frac{1}{a}\int \frac{1}{1+t^2}\,dt$$

Let $t = \operatorname{tg}\theta$. Then $\frac{dt}{d\theta} = \sec^2\theta = 1+\operatorname{tg}^2\theta = 1+t^2$,

$$\int \frac{1}{1+t^2}\,dt = \int \frac{1}{\sec^2\theta}\,\frac{dt}{d\theta}\,d\theta = \int \underbrace{\frac{1}{\sec^2\theta}\cdot\sec^2\theta}_{1}\,d\theta$$

$$= \theta + c = \operatorname{arc\,tg} t + c = \operatorname{arc\,tg}\left(\frac{x}{a}\right) + c$$

$$\therefore \int \frac{dx}{a^2+x^2} = \frac{1}{a}\operatorname{arctg}\left(\frac{x}{a}\right) + c \quad (\text{arb. const.})$$

2. $\displaystyle\int \frac{dt}{\sqrt{1-t^2}}$

as in preceding example

$$= \int \frac{1}{\cos\theta}\,\frac{dt}{d\theta}\,d\theta \qquad \left(\text{with } t = \sin\theta,\ \frac{dt}{d\theta} = \cos\theta\right)$$

$$= \theta + c = \sin^{-1} t + c \quad (\text{arb. const.})$$

Regarding $\sin^{-1} t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

i.e. $\sin^{-1} : \cancel{\mathbb{R}}\ (-1,1) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

3. $\displaystyle\int \frac{dx}{\sqrt{x^2+a}}$

Omitted

$$= \int \frac{1}{\frac{1}{2}(t+a/t)} \frac{dx}{dt}\, dt \quad \left(\begin{array}{l} \text{with } \sqrt{x^2+a} = t-x, \text{ i.e. } x^2+a = (t-x)^2, \\ \text{or } x = \frac{1}{2}\left(t - \frac{a}{t}\right) \end{array} \right)$$

$$= \int \frac{2}{t+a/t} \frac{1}{2}(1+a/t^2)\, dt \quad \left(\text{as } \frac{dx}{dt} = \frac{1}{2}\left(1+\frac{a}{t^2}\right) \right)$$

$$= \int \frac{dt}{t} = \ln\left|\sqrt{x^2+a} + x\right| + c \text{ (arb. const.)}$$

4. $\displaystyle\int \frac{\ln x}{x}\, dx$

$$= \int \frac{t}{x} \frac{dx}{dt}\, dt \qquad \left(\text{with } t = \ln x,\ \frac{dt}{dx} = \frac{1}{x}\right)$$

$$= \int t\, dt = \frac{1}{2}t^2 + c = \frac{1}{2}(\ln x)^2 + c \text{ (arb. const.)}$$

5. $\displaystyle\int \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}$ (where $\alpha < x < \beta$)

Omitted

$$= \int \frac{\frac{dx}{d\theta}\, d\theta}{\sqrt{(\beta-\alpha)(\sin^2\theta)(\beta-\alpha)(\cos^2\theta)}} \quad \left(\text{with } x = \alpha\cos^2\theta + \beta\sin^2\theta\right)$$

$x-\alpha = (\beta-\alpha)\sin^2\theta,\ \beta - x = (\beta-\alpha)\cos^2\theta$

$\left(\frac{dx}{d\theta} = 2(\beta-\alpha)\sin\theta\cos\theta\right)$

$$= \underset{?}{\pm}\int 2\, d\theta = 2\theta + c$$

$$= \underset{?}{(\pm)} 2\, \mathrm{tg}^{-1} \underset{?}{(\pm)} \sqrt{\frac{x-\alpha}{\beta-x}} + c \text{ (arb. const.)}$$

6. $\int x \cos x \, dx$

$= x \sin x - \int 1 \cdot \sin x \, dx \quad \left(\text{with } f(x) = x,\ g'(x) = \cos x,\ f'(x) = 1,\ g(x) = \sin x \right)$

$= x \sin x + \cos x + c$ (arb. const.)

7. $\int \cos^2\theta \, d\theta = \int \frac{1}{2}(1+\cos 2\theta)\, d\theta = \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + c$

$\frac{du}{d\theta}\, d\theta \qquad u = 2\theta$

8. $\int tg^{-1} x \, dx$

try $\theta = tg^{-1}x$ yourself

$= x\, tg^{-1}x - \int x \cdot \frac{1}{1+x^2}\, dx \quad \left(\text{with } f(x) = tg^{-1}x,\ g'(x) = 1,\ g(x) = x,\ f'(x) = \frac{1}{1+x^2} \right)$

$= x\, tg^{-1}x - \frac{1}{2}\int \frac{\frac{du}{dx}}{u} \frac{dx}{du}\, du \quad \left(\text{with } u = 1+x^2,\ \frac{du}{dx} = 2x \right)$

$= x\, tg^{-1}x - \frac{1}{2}\ln(1+x^2) + c$ (arb. const.)

9. $\int e^{ax} \cos bx \, dx$ (assuming $a^2+b^2 \neq 0$) $a, b \in \mathbb{R}$

$= \frac{1}{a}\left[e^{ax}\cos bx + b\int e^{ax} \sin bx \, dx \right] \quad \left(\text{with } g'(x) = e^{ax},\ g(x) = \frac{1}{a}e^{ax},\ f(x) = \cos bx,\ f'(x) = -b\sin bx \right)$

(assuming $a \neq 0$ temporally)

$= \frac{1}{a}e^{ax}\cos bx + \frac{b}{a}\left[\frac{1}{a}e^{ax}\sin bx - \frac{b}{a}\int e^{ax}\cos bx \, dx \right] \quad \left(\text{with } g'(x) = e^{ax},\ g(x) = \frac{1}{a}e^{ax},\ h(x) = \sin bx,\ h'(x) = b\cos bx \right)$

$= \frac{1}{a}e^{ax}\cos bx + \frac{b}{a^2}e^{ax}\sin bx - \frac{b^2}{a^2}\int e^{ax}\cos bx \, dx$,

$\therefore \int e^{ax}\cos bx \, dx = \frac{1}{a^2+b^2}(a\cos bx + b\sin bx)\, e^{ax} + c$ (arb. const.)

10. Similarly,

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx) + c$$

11. $\displaystyle\int \frac{dx}{x^2-a^2}$ (where $a \neq 0$)

$$= \int \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right) dx$$

$$= \frac{1}{2a}\left[\int \frac{dx}{x-a} - \int \frac{dx}{x+a}\right]$$

$$= \frac{1}{2a}\left(\ln|x-a| - \ln|x+a|\right) + c$$

$$= \frac{1}{2a} \ln \left|\frac{x-a}{x+a}\right| + c.$$

This last example is an example of <u>integration by partial fraction expansions</u>:

$$\frac{1}{x^2-a^2} = \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right).$$

12. $\int \sec\theta \, d\theta$

(in class)

$$\because \sec\theta = \frac{1}{\cos\theta} \overset{\downarrow}{\doteq} \frac{\cos\theta}{\cos^2\theta} = \frac{\cos\theta}{1-\sin^2\theta}$$

$$= \left(\frac{1/2}{1-\sin\theta} + \frac{1/2}{1+\sin\theta}\right)\cos\theta \,.$$

$\Big[$ i.e. $\sec\theta\, d\theta = \frac{1}{1-u^2}du$, where $u = \sin\theta$

$$\therefore \int \sec\theta\, d\theta = \frac{1}{2}\left[\int \frac{1}{1-\sin\theta}\frac{d(\sin\theta)}{d\theta}d\theta + \int \frac{1}{1+\sin\theta}\frac{d(\sin\theta)}{d\theta}d\theta\right]$$

$$= \frac{1}{2}\Big[-\ln|1-\sin\theta| + \ln|1+\sin\theta|\Big] + \underset{\substack{\uparrow \\ \text{constant}}}{c}$$

$$= \ln\left|\frac{1+\sin\theta}{1-\sin\theta}\right|^{1/2} + c$$

$$= \ln\left[\frac{(1+\sin\theta)^2}{1-\sin^2\theta}\right]^{1/2} + c$$

$$= \ln\left|\frac{1+\sin\theta}{\cos\theta}\right| + c$$

$$= \ln|\sec\theta + \tan\theta| + c,$$

where c is an arbitrary constant.

Partial Fractions

e.g. of degree 3 > degree of the donominator (x^2-2x-3)

$$\frac{7x^3+3x^2+6}{x^2-2x-3} = 7x+17+\frac{55x+57}{x^2-2x-3}$$

of degree < that of the denominator,

$$\frac{55x+57}{x^2-2x-3} = \frac{55x+57}{(x-3)(x+1)}$$

distinct factors

Suppose $\dfrac{55x+57}{(x-3)(x+1)} = \dfrac{A}{x-3} + \dfrac{B}{x+1}$, where A, B are constants to be determined. Then

polynomials of degree < that of denominator

$$55x+57 = A(x+1) + B(x-3) \quad \cdots (1)$$

$(A+B)x + (A-3B)$

$\begin{cases} A+B=55 \\ A-3B=57 \end{cases}$

To determine B,
Put $x=-1$ into (1), we get:

$$B = -\frac{1}{4}(-55+57), \quad B = -\frac{1}{2}.$$

To determine A,
Put $x=3$ into (1), we get:

$$A = \frac{1}{4}(55\cdot 3+57) = 55.5$$

$$\therefore \quad \frac{55x+57}{x^2-2x-3} = \frac{55.5}{x-3} - \frac{1/2}{x+1}$$

We check that is true for all $x \neq -1, 3$

$$\therefore \int \frac{7x^3+3x^2+6}{x^2-2x-3}\,dx = 7\int x\,dx + 17\int dx + 55.5\int\frac{dx}{x-3} - \frac{1}{2}\int\frac{dx}{x+1}$$

$$= \frac{7}{2}x^2 + 17x + 55.5\ln|x-3| - \frac{1}{2}\ln|x+1| + C.$$

Another example: $\dfrac{2x^2+2x+13}{x^5-2x^4+2x^3-4x^2+x-2}$

$$= \frac{2x^2+2x+13}{(x-2)\underbrace{(x^2+1)^{2}}_{\text{repeated factor}}}$$

Suppose

$$\frac{2x^2+2x+13}{(x-2)(x^2+1)^2} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}, \text{ where}$$

A, B, C, D and E are constants to be determined.

Then $2x^2+2x+13$

$$= A(x^2+1)^2 + (Bx+C)(x-2)(x^2+1) + (Dx+E)(x-2) \quad \cdots(1)$$

Put $x=2$ into (1), we get: $25A = 25$, $\therefore A = 1$

Putting $A=1$ into (1), we get:

$$-x^4+2x+12 = (x-2)\left[(Bx+C)(x^2+1) + Dx+E\right] \quad \cdots(2)$$

Since $-x^4+2x+12 = (x-2)(-x^3-2x^2-4x-6)$,

from (2) we get:

$$-x^3-2x^2-4x-6 = (Bx+C)(x^2+1) + Dx+E \quad \cdots(3)$$

for $x \neq 2$; by continuity, (3) holds for all x,

i.e. $$-x^3-2x^2-4x-6 = Bx^3+Cx^2+(B+D)x+(C+E) \quad \cdots(4)$$

Comparing corresponding coefficients in (4), we get:

$$B=-1,\ C=-2,\ B+D=-4,\ C+E=-6$$

$\therefore$ $B=-1$, $C=-2$, $D=-3$, $E=-4$. One checks easily that

$$\frac{2x^2+2x+13}{(x-2)(x^2+1)^2} = \frac{1}{x-2} - \frac{x+2}{x^2+1} - \frac{3x+4}{(x^2+1)^2} \quad \text{for all } x\ (\neq 2).$$