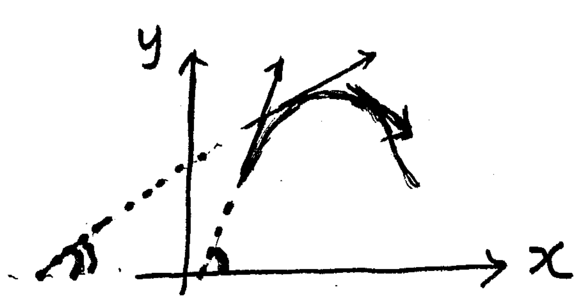
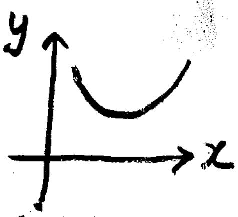
§ Convexity & Points of Inflexion

Let $f:(a,b)\to\mathbb{R}$, and $f''(c)$ is well-defined (exists) and <0 for all $c\in(a,b)$. Then $f'(x)$ is decreasing on (a,b), which implies that the angle between the tangent (to the graph of f) at $(c,f(c))$ and the positive x-axis decreases as c moves from left to right in (a,b). The graph is convex up upward: concave down

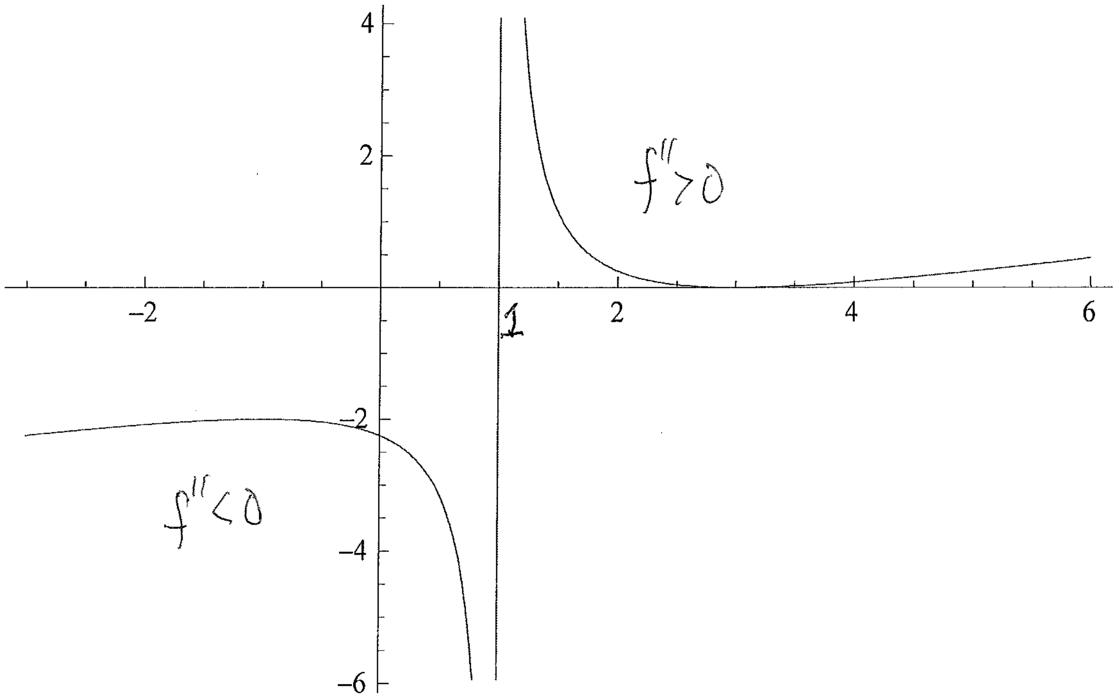


Similarly, if $f''(c)>0$ for all $c\in(a,b)$, then the graph of f is convex downward: concave up



$c\in(a,b)$ is called a point of inflexion (of f) if $f''(c)=0$ and $f''(x)$ changes its sign when x moves from the left of c to the right of c (suff. close to c).

The graph of (x-3)^2/[4(x-1)]

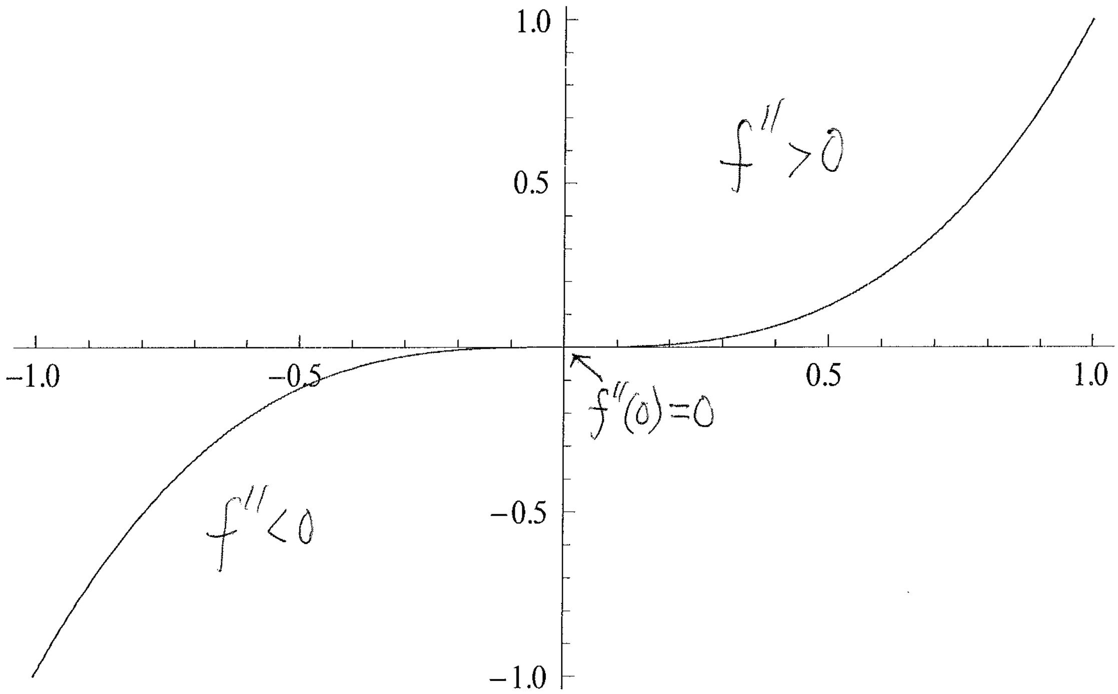


The function (or, graph) has no points of inflexion.

The graph of $f(x)=x^3$, x in [-1, 1].

$f''(x)=6x$, $f''(x)<0$, if $x<0$, $f''(0)=0$, $f''(x).0$, if $x>0$.

$f''(x)=6x$



0 is a point of inflexion.

§ Graphing Strategy

$y=f(x)$

In order to draw the graph of a function f, we follow the following steps:

1) Determine the domain of f.
2) Determine the symmetries of the graph, if any. (symmetric about the y-axis? the origin? ...)
3) Find the intercepts (the intersections of the graph and the x-axis, and the y-axis).
4) Solve $f'(x)=0$, find the local extrema (maxima / minima) if exist, and determine the increasing/decreasing intervals.

skipped 5) Solve $f''(x)=0$, and determine points of inflexion, & convexity (convex upward/downward)

6) Find the asymptotes, if any:

vertical asymptotes $x=c$: $\lim_{x\to c^+} f(x) = \infty$, or $-\infty$

$\lim_{x\to c^-} f(x) = \infty$, or $-\infty$.

horizontal asymptotes $y=d$: (i) $f(x) > d$ for x suff. large and $\lim_{x\to\infty} f(x) = d$

or (ii) $f(x) < d$ for x suff. large, $\lim f(x) = d$ } or (iii), (iv) for $x \to -\infty$.

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oblique asymptotes $y = ax + b$ $(a \neq 0)$

such that $\lim_{x\to\infty}(f(x)-ax-b)=0$,

or $\lim_{x\to-\infty}(f(x)-ax-b)=0$,

i.e. $\begin{cases} a = \lim_{x\to\infty}\frac{f(x)}{x}, \\ b = \lim_{x\to\infty}(f(x)-ax) \end{cases}$ or $\begin{cases} a = \lim_{x\to-\infty}\frac{f(x)}{x}, \\ b = \lim_{x\to-\infty}(f(x)-ax) \end{cases}$

$f(x) = \underline{ax+b} + $

Ex. Sketch the graph of $y = f(x) = \frac{(x-3)^2}{4(x-1)}$.

Soln The domain of f is $\mathbb{R}\setminus\{1\}$. It is continuous on $(1,\infty)$ and on $(-\infty,1)$.

The y-intercept is: $f(0) = -9/4$ (i.e. $(0,-9/4)$)

The x-intercept is: $(3,0)$

By examining $f'(x) = \frac{(x-3)(x+1)}{4(x-1)^2}$, $x\in\mathbb{R}\setminus\{1\}$,

We see that f has a local minimum at $x=3$, $f(3)=0$

& " " maximum at $x=-1$, $f(-1)=-2$

By examining $f''(x) = \frac{2}{(x-1)^3}$, we see that

f is convex downward on $(1,\infty)$

& " upward on $(-\infty,1)$

The graph as a vertical asymptote: $x=1$

and $\lim_{x\to1^+}f(x)=\infty$, $\lim_{x\to1^-}f(x)=-\infty$.

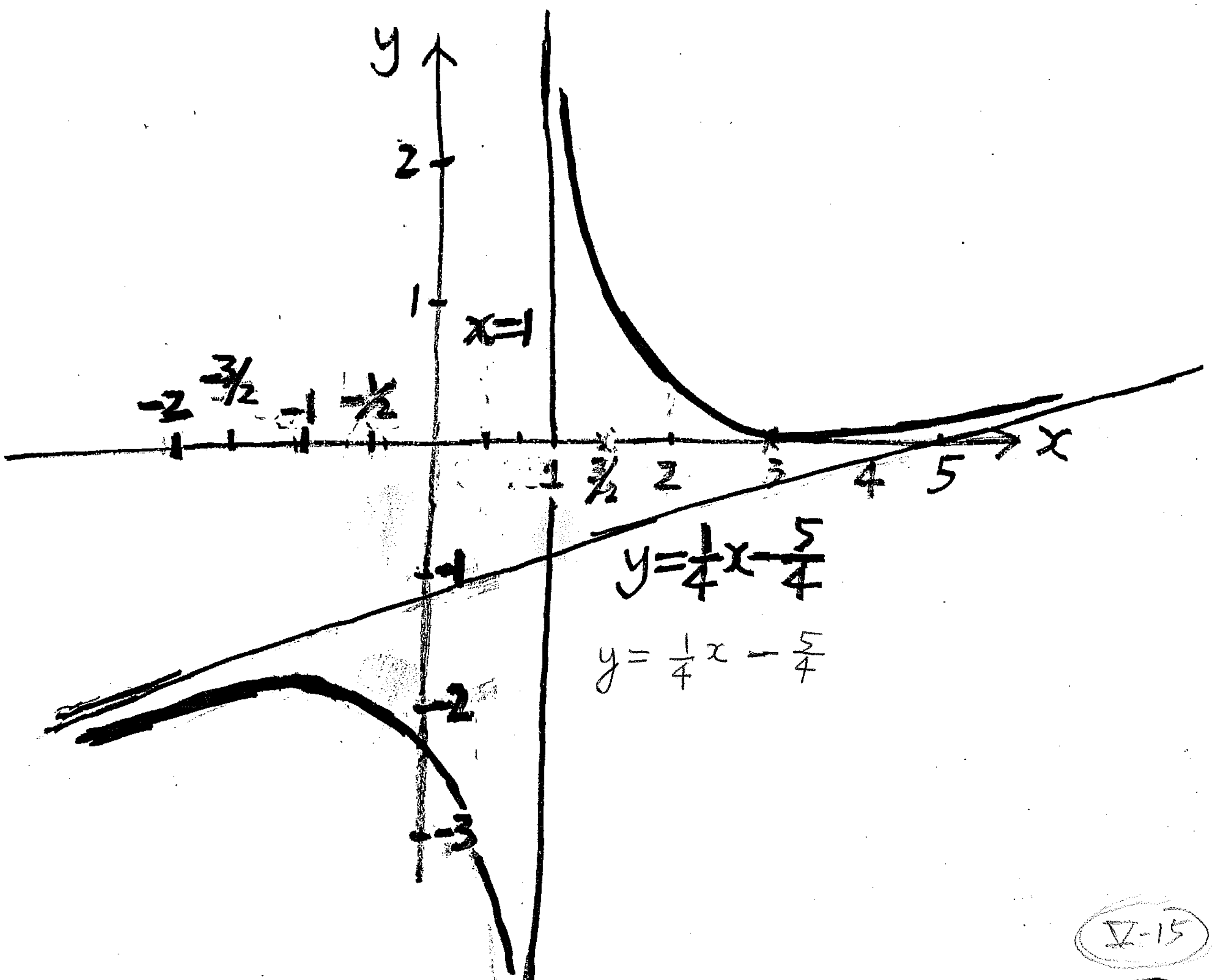
Because $f(x)=\frac{1}{4}x-\frac{5}{4}+\frac{1}{x-1}$,

the graph has an oblique asymptote $y=\frac{1}{4}x-\frac{5}{4}$.

From the above observation, together with the following table

x	4	1.5	0.5	−0.5	−1.5	−2
y	0.1	1.1	−3.1	−2.04	−2.03	−2.08

we sketch the graph of $y=f(x)$ as follows



Let $f: (a,b) \to \mathbb{R}$. f is said to be <u>convex</u>

if for all $x_1, x_2 \in (a,b)$,

$$f\left(\frac{x_1+x_2}{2}\right) \leq \frac{1}{2}\left(f(x_1)+f(x_2)\right).$$

i.e. f (mid-pt. of x_1, x_2) is not higher (greater)

than the mid-pt. of $f(x_1)$ and $f(x_2)$.

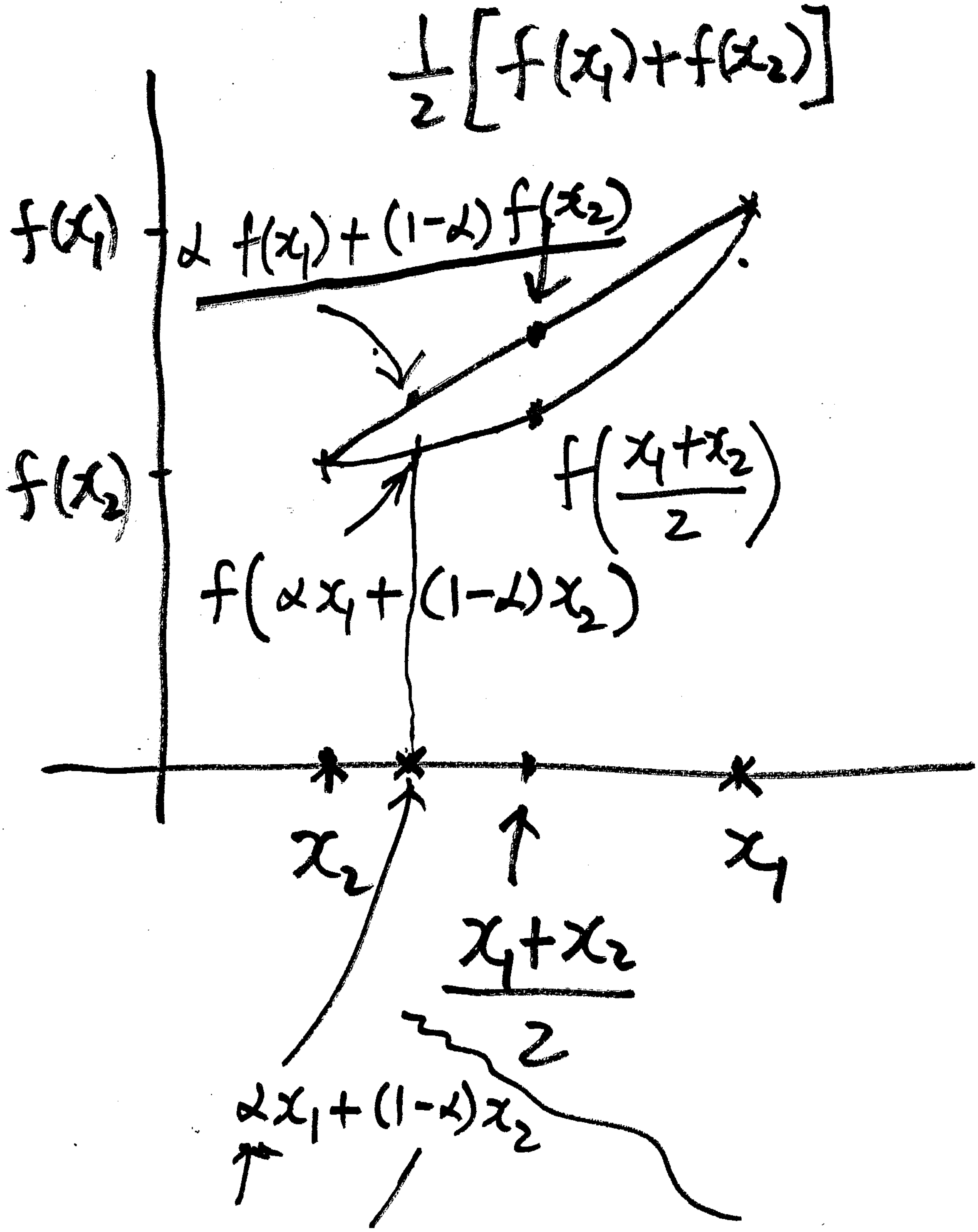
<u>Thm</u> Suppose $f: (a,b) \to \mathbb{R}$ is continuous on (a,b).

(i) If f is convex, then for all $\alpha \in (0,1)$, $x_1, x_2 \in (a,b)$,

$$f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha f(x_1) + (1-\alpha) f(x_2).$$

(ii) Suppose f'' is well-defined on (a,b). Then

f is convex *iff* $f''(x) \geq 0$ for all $x \in (a,b)$.


$\frac{1}{2}[f(x_1)+f(x_2)]$
$f(x_1)$
$\alpha f(x_1)+(1-\alpha)f(x_2)$
$f(x_2)$
$f\left(\frac{x_1+x_2}{2}\right)$
$f(\alpha x_1+(1-\alpha)x_2)$
x_2
x_1
$\frac{x_1+x_2}{2}$
$\alpha x_1+(1-\alpha)x_2$