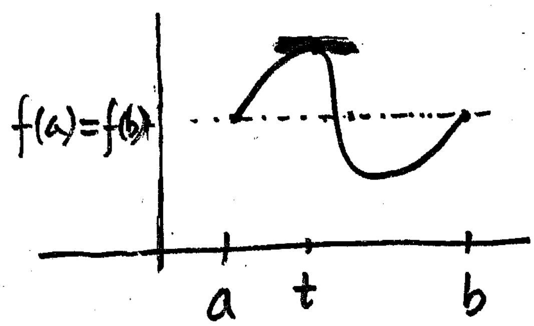
§ Mean Value Thm & Extrema

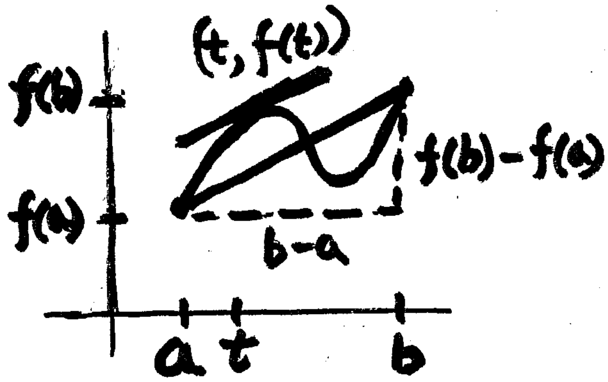
Thm (Mean Value) Suppose $a<b$ in $\mathbb{R}$. Suppose $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b).

Then there exists a $t\in(a,b)$ such that

$$f'(t)=\frac{f(b)-f(a)}{b-a}.$$

The following special case is known as Roll's thm:
Assume conditions of the above thm plus $f(b)=f(a)$.
Then there exists a $t\in(a,b)$ such that $f'(t)=0$.





Corollary 1. Assume conditions of the Mean Value thm, and suppose $f'(x)\geq 0$ ($>$) for all $x\in(a,b)$.
Then f is (strictly) increasing on $[a,b]$, i.e.

$$a\leq x_1<x_2\leq b \Rightarrow f(x_1)\leq f(x_2) \quad (<)$$

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Corollary 2 Assume conditions of the mean value thm and suppose $f'(x) \leq 0$ (strictly) ($<$) for all $x \in (a,b)$. Then f is decreasing on $[a,b]$, i.e.

$$a \leq x_1 < x_2 \leq b \Rightarrow f(x_1) \geq f(x_2) \quad (>).$$

Ex. 1 Let $f(x) = x^2$, $x \in \mathbb{R}$. Then

$$f'(x) = 2x = \begin{cases} >0 & \text{if } x>0, \\ =0 & \text{if } x=0, \\ <0 & \text{if } x<0. \end{cases}$$

So f is strictly increasing on $[0,\infty)$, & strictly decreasing on $(-\infty, 0]$.

Ex. 2 Let $f(x) = \sin x - x\cos x$, $x \in [0, \frac{\pi}{2}]$.

Then $f'(x) = x \sin x > 0$ for all $x \in (0, \frac{\pi}{2})$.

Thus f is strictly increasing on $[0, \frac{\pi}{2}]$. So for each $x \in (0, \frac{\pi}{2}]$, $f(x) > f(0) = 0$ i.e.

$\sin x - x\cos x > 0$ i.e. $\frac{\sin x}{x} > \cos x$.

Similarly we can prove that for all $x \in (0, \frac{\pi}{2})$,

$$\frac{\sin x}{x} < \frac{1}{\cos x}.$$

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Ex.3. Let $g: [-1,1] \to \mathbb{R}: x \mapsto 3x^2+4x-7$.

Then for all $x \in (-1,1)$,

$$g'(x) = 6x+4 = 2(3x+2) = \begin{cases} <0, & x \in (-1,-\frac{2}{3}) \\ =0, & x = -\frac{2}{3}, \\ >0, & x \in (-\frac{2}{3}, 1). \end{cases}$$

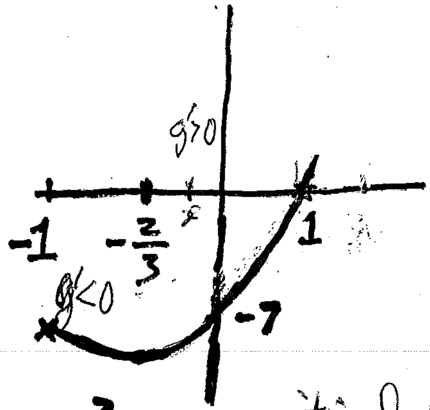
Hence g is (strictly) decreasing on $[-1,-\frac{2}{3}]$,

and " " increasing on $[-\frac{2}{3}, 1]$.

And $\underbrace{\max_{[-1,1]} g}_{} = \max\{g(1), g(-1)\} = g(1) = 0$,

this denotes the maximum value of $g(x)$, $x \in [-1,1]$.

$\min_{[-1,1]} g = g(-\frac{2}{3}) = -25/3$



Method 2 For $x \in (-1,1)$

$$g'(x) = 2(3x+2) = 0 \Leftrightarrow x = -\frac{2}{3}.$$

critical points
critical values

Then $\max_{[-1,1]} g = \max\{g(-\frac{2}{3}), g(1), g(-1)\}$

$= \max\{-\frac{25}{3}, 0, -8\} = 0$,

$\min_{[-1,1]} g = \min\{g(-\frac{2}{3}), g(1), g(-1)\} = -\frac{25}{3}$

Reasons: Because g is continuous on $[-1,1]$,

there exists $d \in [-1,1]$ such that $g(d) = \max_{[-1,1]} g$.

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IV.7

We have 3 alternatives (possibilities): $d \in (-1,1)$, $d=-1$, or $d=1$.

If $d \in (-1,1)$, then for $h>0$ and sufficiently small,

$$g(d) \geq g(d+h), \quad g(d) \geq g(d-h),$$

so

$$\frac{g(d+h)-g(d)}{h} \leq 0$$

$$0 \leq \frac{g(d-h)-g(d)}{-h}.$$

Because $g'(d) = \lim\limits_{h \to 0^+} \dfrac{g(d+h)-g(d)}{h} = \lim\limits_{-h \to 0^-} \dfrac{g(d-h)-g(d)}{-h}$

we obtain:

$$0 \leq g'(d) \leq 0,$$

i.e. $\boxed{g'(d)=0}$,

i.e. d is a solution of the equation $g'(x)=0$

i.e. $d = -\frac{2}{3}$ in the present case.

Therefore,

$$\max_{[-1,1]} g = \max\left\{ g(-1), g(1), g\left(-\tfrac{2}{3}\right) \right\}.$$

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Similarly, we have a similar result for minimum

i.e. $\min\limits_{[-1,1]} g = \min\{ g(x) :$ x is an end point of the domain (interval) of g, or $g'(x)=0 \}$.

Note that the above reasons are applicable to any function f which is continuous on $[a,b]$ and differentiable on (a,b) (not just our particular function g), so

$$\max_{[a,b]} f = \max\{f(x) : x = a, \text{ or } b, \text{ or } f'(x)=0\}$$

$$\min_{[a,b]} f = \min\{f(x) : x = a, \text{ or } b, \text{ or } f'(x)=0\}.$$

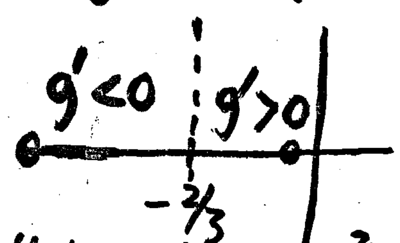
Let $c \in [a,b]$. We say that $f(c)$ is a relative local maximum mimum if for $|h|$ sufficiently small and $c+h \in [a,b]$ (for $c=a$, h must be >0; for $c=b$, h must be <0): $f(c) \leqslant f(c+h)$, | $f(c) \geqslant f(c+h)$

i.e. if $x \in [a,b]$ and x is sufficiently close to c,

$$f(c) \leqslant f(x).$$

For our particular function g above, $g'(x) = 2(3x+2)$ 14 V-8 IV-9

$$\begin{cases} >0 \text{ if } x > -\frac{2}{3} \text{ (\& suff. close to } -\frac{2}{3}) \\ <0 \text{ if } x < -\frac{2}{3} \text{ (\& suff. close to } -\frac{2}{3}). \end{cases}$$



$g \downarrow$ (falls to) $g(-2/3)$ on suff. close, left to $-2/3$
$g \uparrow$ (rises from) $g(-2/3)$ on .. .., right to $-2/3$
$\Big\} \Rightarrow g(-\frac{2}{3})$ is a local minimum.

Such argument is also applicable in many cases. We have similar consideration for a local maximum.

Also, in general, if $f(c)$ is a local maximum or local mimum of $f: [a,b] \to \mathbb{R}$, if $c \in (a,b)$, and if $f'(c)$ exists (i.e. f has a derivative at c), then $f'(c) = 0$. So, to find <u>local maxima and local minima</u> of $f: [a,b] \to \mathbb{R}$, we can follow the following procedure, <u>assuming f is differentiable on (a,b)</u>:

1. Solve $f'(x) = 0$, let the solutions be $x_1, x_2, \cdots, x_n$.
2. Examine the behavior of $f(x)$ for x (suff.) close to each x_k, $k = 1, 2, \cdots, n$, and a, b (the end points).

   If possible, consider the signs of $f'(x)$.
   If $f'(x)$ changes from $\pm$ to $\mp$ as x moves (close to x_k) from left of x_k to right of x_k, then $f(x_k)$ is a local $\overset{\text{maximum}}{\text{minimum}}$.

   Similar criterion applies to the end points (just one side)

   $f(a)$ $\overset{\text{loc.}}{\text{min.}}$ if $\underset{a}{\vdash}\!\overset{f'>0}{\text{———}}\!\circ$; $f(a)$ $\overset{\text{loc.}}{\text{max}}$ if $\underset{a}{\vdash}\!\overset{f'<0}{\text{———}}\!\circ$

Ex. The above $g: [-1,1] \to \mathbb{R}: x \mapsto 3x^2+4x-7$ admits a local minimum at $-\frac{2}{3}$, i.e. $g(-\frac{2}{3})$ is a local minimum. It admits local maxima at -1 and 1, i.e. $g(-1)$ and $g(1)$ are local maxima.

§ Higher derivatives, Extrema

f' is a function itself, so we may consider its derivative $(f')'$, which is denoted by f'', $f^{(2)}$, or $\frac{d^2f}{dx^2}$, and called the second derivative of f.

Similarly we have f''', $f^{(3)}$, or $\frac{d^3f}{dx^3}$, etc.

Ex.1 Let $f(x)=x$, $x\in\mathbb{R}$. Then $f'(x)=1$, and $f''(x)=(f')'(x)=(1)'=0$, $x\in\mathbb{R}$.

Ex.2 Let $f(x)=\sin x$, $x\in\mathbb{R}$. Then $f'(x)=\cos x$, $f''(x)=-\cos x$, $f'''(x)=-\sin x$, $x\in\mathbb{R}$.

Suppose $f:[a,b]\to\mathbb{R}$, $c\in(a,b)$, $f'(c)=0$ and $f''(c)>0$. Then for $h>0$ sufficiently small,

$$\frac{f'(c+h)}{h}=\frac{f'(c+h)-f'(c)}{h}\approx f''(c)>0,$$

so that $\quad \dfrac{f'(c+h)}{h}>0$, hence $f'(c+h)>0$.

That is,

$$f'(c+h)>0 \text{ for } h>0 \text{ suff. small.}$$

Similarly we have

$$f'(c-h)<0 \text{ for } h>0 \text{ suff. small.}$$

Thus, if $f'(c)=0$ and $f''(c)>0$, then f is increasing on the right of c, and is decreasing on the left of c, so f has a local minimum at c.

<u>Thm</u>. Suppose $f:[a,b]\to\mathbb{R}$, $c\in(a,b)$, $f'(c)=0$.

(i) If $f''(c)>0$, then $f(c)$ is a local minimum.

(ii) If $f''(c)<0$, then $f(c)$ is a local maximum.