§ Sequences, Limits

Sequence $\quad (a_n)$, $(a_n)_{n\in\mathbb{N}}$, $\{a_n\}$

where each a_n is a (real) number,

e.g. $\quad \left(\frac{1}{n}\right)$, $\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\right)$

$\left((-1)^n\right)$, $(-1, 1, -1, 1, \cdots)$

$(\sin n)$, $(\sin 1 = 0.17,\ \sin 2 = 0.35,\ \sin 3 = 0.05,\ -0.07, \cdots$

A sequence can be regarded as a map $f : \mathbb{N} \to \mathbb{R}$, where $f(n) = a_n$.

Observe that

(1) $\underset{a_n}{\frac{1}{n}}$ {can be made / is} arbitrarily close to 0 / ℓ with n sufficiently large, Given $\varepsilon > 0$, $|a_n - \ell| < \varepsilon$ when n is large enough

e.g. $\left|\frac{1}{n} - 0\right| < 0.1$ if $n > \frac{1}{0.1} = 10$;

$\left|\frac{1}{n} - 0\right| < 10^{-3}$ if $n > 10^3$;

etc. $\quad$ (ε–N formulation) converges

We say: "the sequence $\left(\frac{1}{n}\right)$ tends to 0 (as n tends to infinity)"

and write: " $\lim\limits_{n\to\infty} \frac{1}{n} = 0$ "

Similarly

$$\lim_{n\to\infty} \frac{3}{5n-7} = 0, \quad \lim_{n\to\infty} \frac{2n+1}{n^2+5} = \lim_{n\to\infty} \frac{2+\frac{1}{n}}{n+\frac{5}{n}}$$

$$= 0 \quad \text{as} \ldots\ldots$$

$$\lim_{n\to\infty} \frac{\sin n}{(0.5)n} = 0 \quad \text{as} \ldots$$

(2) n^2 $\left\{\begin{array}{l}\text{is very large} \\ \text{can be made arbitrarily large}\end{array}\right\}$ with n sufficiently large.

e.g. $n^2 > 1376$ if $n > 400$;

$n^2 > 10^7$ if $n > 10^4$;

etc.

diverges to ∞

We say: "the sequence (n^2) tends to infinity (as n tends to infinity)"

and write " $\lim\limits_{n\to\infty} n^2 = \infty$ "

Similarly

$$\lim_{n\to\infty} \frac{n^2+1}{2n+3} = \lim_{n\to\infty} \frac{(n^2+1)/n}{(2n+3)/n}$$

$$= \lim_{n\to\infty} \frac{n+\frac{1}{n}}{2+\frac{3}{n}} = \infty$$

$$\left(\text{as } \frac{n+\frac{1}{n}}{2+\frac{3}{n}} > \frac{n}{6} > 10^{+p} \quad \text{if } n > 6\cdot 10^{p}, \text{ where } p\in\mathbb{N}. \right),$$

$p = 7,\ 10,\ 1000, \ldots$

$$\lim_{n\to\infty} \frac{n}{|\sin n|} = \infty .$$

(3) $-\sqrt{n}$ can be made arbitrarily small with n sufficiently large

e.g. $-\sqrt{n} < -1597$ if $n > (1597)^2$;

$-\sqrt{n} < -10^8$ if $n > 10^{16}$;

etc.

We say: "$(-\sqrt{n})$ tends to negative infinity"

and write "$\lim_{n\to\infty} (-\sqrt{n}) = -\infty$".

(4)

$(-1)^n$ does not come close } to any number, or ± infinity
no matter how large n is

e.g.
among all $n > 10^8$,

there are odd n's (10^8+1, say): $(-1)^n = -1$

" even n's (10^8+2, say): $(-1)^n = 1$.

We say: " $\{(-1)^n\}$ does not tend to any number or ± infinity "

and write: " $\lim_{n\to\infty} (-1)^n$ does not exist "
as a real number
or ± infinity

Similarly

$\lim_{n\to\infty} (-1)^n \sqrt{n}$ does not exist, because ...

$\lim_{n\to\infty} \dfrac{1}{\sin\left(\frac{(-1)^n}{n}\right)}$ does not exist, because ...

Thm 1 Suppose $\lim_{n\to\infty} a_n = \ell$, $\lim_{n\to\infty} b_n = b$.

(0) If $\lim_{n\to\infty} a_n = p$, then $\ell = p$. (uniqueness of limit.)

(1) Suppose $\ell, b \in \mathbb{R}$. Then

(i) $\lim_{n\to\infty} |a_n| = |\ell|$.

(ii) $\lim_{n\to\infty} (a_n + b_n) = \ell + b = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$

(iii) $\lim_{n\to\infty} (k a_n) = k\ell = k \lim_{n\to\infty} a_n$, where $k \in \mathbb{R}$,

(iv) $\lim_{n\to\infty} (a_n b_n) = \ell b = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n)$,

(v) $\lim_{n\to\infty} \frac{b_n}{a_n} = \frac{b}{\ell} = \lim_{n\to\infty} b_n / \lim_{n\to\infty} a_n$, provided $\underbrace{\ell}_{\lim_{n\to\infty} a_n} \neq 0$.

II-5-a

(2) $\lim_{n\to\infty} \frac{c_n}{a_n} = 0$ if $\overset{\lim_{n\to\infty} a_n}{\ell} = \infty$ or $-\infty$, and

if $|c_n| \leq k$ for all $n \in \mathbb{N}$ (where k is a fixed constant number).

(3) $\lim_{n\to\infty} \frac{1}{a_n} = \infty$ if $\overset{\lim_{n\to\infty} a_n}{\ell} = 0$ and $a_n > 0$ for sufficiently large n;

$\lim_{n\to\infty} \frac{1}{a_n} = -\infty$ if $\overset{\lim_{n\to\infty} a_n}{\ell} = 0$ and $a_n < 0$ for sufficiently large n

(4) If $a_n \leq b_n$ for sufficiently large n, then $\ell \leq b$ (where it is assumed that $\ell, b \in \mathbb{R}$).

(5) If $\ell < c$ in $\mathbb{R}$, then $a_n < c$ for sufficiently large n.

(6) If $(n_k)_{k\in\mathbb{N}}$ is an strictly increasing sequence in $\mathbb{N}$, i.e. if $n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots$,
then $\lim_{k\to\infty} a_{n_k} = \ell$.

II-5-b

Thm 2 Suppose $a_n \leq b_n \leq c_n$ for n sufficiently large, and suppose $\lim_{n\to\infty} a_n = \ell$, $\lim_{n\to\infty} c_n = \ell$, where $\ell \in \mathbb{R} \cup \{\infty, -\infty\}$. Then

$$\lim_{n\to\infty} b_n = \ell.$$

(This is called the **sandwich theorem**.)

Thm 3 Suppose there is a constant k such that

$a_n \leq a_{n+1} \leq k$ for sufficiently large n.

Then (a_n) tends to a certain number $\ell \leq k$

i.e. (there exists an $\ell \in (-\infty, k]$ such that)

$$\lim_{n\to\infty} a_n = \ell.$$

Thm 3' Suppose there is a constant r such that

$a_n \geq a_{n+1} \geq r$ for sufficiently large n.

Then there exists $t \in [r, \infty)$ such that

$$\lim_{n\to\infty} a_n = t. \qquad t \geqslant r$$

Ex.1. Because $\frac{1}{n} \geq \frac{1}{n+1} \geq 0$ for all n,

there exists $\ell \in \mathbb{R}$ such that $\lim_{n\to\infty} \frac{1}{n} = \ell$.

Since $2n \in \mathbb{N}$ whenever $n \in \mathbb{N}$, we have $\lim_{n\to\infty} \frac{1}{2n} = \ell$.

On the otherhand, $\lim_{n\to\infty} \frac{1}{2n} = \lim_{n\to\infty} \frac{1}{2}\left(\frac{1}{n}\right) = \frac{1}{2} \lim_{n\to\infty} \frac{1}{n} = \frac{1}{2}\ell$.

$\therefore \ \ell = \frac{1}{2}\ell$, $\quad (1-\frac{1}{2})\ell = 0$, $\ \therefore \ \ell = 0$ i.e.

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

Ex. 2. $\lim_{n\to\infty} \frac{5-3n}{n^2}$

$$= \lim_{n\to\infty} \frac{5}{n^2} - 3\left(\lim_{n\to\infty} \frac{n}{n^2}\right) = 5\left(\lim_{n\to\infty} \frac{1}{n}\right)^2 - 3\left(\lim_{n\to\infty} \frac{1}{n}\right) \quad \text{(by Thm 1)}$$

$$= 5(0)^2 - 3(0) = 0.$$

Alternatively, because

$$-\frac{3}{n} < \frac{5-3n}{n^2} < \frac{1}{n} \quad \text{for } n > 5,$$

and because $\lim_{n\to\infty}\left(\frac{1}{n}\right) = 0 = \lim_{n\to\infty}\left(-\frac{3}{n}\right)$, we have

(by Thm.2):

$$\lim_{n\to\infty} \frac{5-3n}{n^2} = 0.$$

Ex. 3. The sequence $\left(\left(1+\frac{1}{n}\right)^n\right)_{n\in\mathbb{N}}$ converges to a real number (which is denoted by e), called the exponential (number).

Skip this this week.

Graph

Pf. First, we can show that for all $n \in \mathbb{N}$, $\left(1+\frac{1}{n}\right)^n < 3$.

For example,

$$\left(1+\frac{1}{2}\right)^2 = 1 + 2(1)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 \quad (\text{by the binomial thm})$$
$$< 1+1+1 = 3,$$
$$\left(1+\frac{1}{3}\right)^3 = 1 + 3\left(\frac{1}{3}\right) + 3\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 \quad (\text{by the binomial thm})$$
$$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} < 3.$$

This can be ignored, if you like.

In general,

$$\left(1+\frac{1}{n}\right)^n = 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{2}\frac{1}{n^2} + \frac{1}{3(2)}\frac{n(n-1)(n-2)}{n^3} + \cdots + \frac{1}{n^n}$$
$$= 1 + 1 + \frac{1}{2}\left(1-\frac{1}{n}\right) + \frac{1}{3(2)}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots + \frac{1}{n^n}$$
$$< 1 + 1 + \underbrace{\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}} = 2 + \underbrace{\left(\frac{1}{2} - \frac{1}{2^n}\right)\Big/\frac{1}{2}}$$
$$< 3.$$

Secondly, we can show that for all $n \in \mathbb{N}$,

$$\left(1+\frac{1}{n+1}\right)^{n+1} > \left(1+\frac{1}{n}\right)^n.$$

For example,

$$\left(1+\frac{1}{3}\right)^3 = 1 + 3\left(\frac{1}{3}\right) + \frac{1}{2}\frac{3(2)}{3(3)} + \left(\frac{1}{3}\right)^3 = 1 + 1 + \frac{1}{2}\left(1-\frac{1}{3}\right) + \left(\frac{1}{3}\right)^3$$
$$> 1 + 2\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(1-\frac{1}{2}\right) = \left(1+\frac{1}{2}\right)^2$$
$$\left(1+\frac{1}{4}\right)^4 = 1 + 4\left(\frac{1}{4}\right) + \frac{1}{2}\frac{4(3)}{4(4)} + \frac{1}{3(2)}\frac{4(3)(2)}{4(4)(4)} + \left(\frac{1}{4}\right)^4$$
$$> 1 + 3\left(\frac{1}{3}\right) + \frac{1}{2}\left(1-\frac{1}{3}\right) + \frac{1}{3(2)}\left(1-\frac{1}{3}\right)\left(1-\frac{2}{3}\right) = \left(1+\frac{1}{3}\right)^3$$

skip this this week

This can be ignored, if you like.

In general,

$$\left(1+\frac{1}{n+1}\right)^{n+1} = 1 + (n+1)\left(\frac{1}{n+1}\right) + \frac{1}{2}\frac{(n+1)(n)}{(n+1)^2} + \frac{1}{3(2)}\frac{(n+1)n(n-1)}{(n+1)^3} + \cdots + \frac{n+1}{(n+1)^n} + \frac{1}{(n+1)^{n+1}}$$

$$= 2 + \frac{1}{2}\left(1-\frac{1}{n+1}\right) + \frac{1}{3(2)}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right) + \cdots + \frac{n+1}{(n+1)^n} + \frac{1}{(n+1)^{n+1}}$$

$$> 2 + \frac{1}{2}\left(1-\frac{1}{n}\right) + \frac{1}{3(2)}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots + \frac{1}{n^n}$$

$$= \left(1+\frac{1}{n}\right)^n.$$

$$\frac{n+1}{(n+1)^n} = \frac{(n+1)!}{(n!)(n+1)^n} = \frac{1}{n!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{n-1}{n+1}\right)$$

$$> \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)$$

$$= \frac{1}{n^n}.$$

Therefore, the desired conclusion follows from Thm. 3.

(Note that $2 < e < 3$ from the above argument; in fact, $e \simeq 2.718$.)

Thm 4 If $\lim_{n\to\infty} a_{2n} = \ell = \lim_{n\to\infty} a_{2n-1}$, where $\ell \in \mathbb{R} \cup \{-\infty, \infty\}$, then $\lim_{n\to\infty} a_n = \ell$.

Ex.4 Let $x \in [0,1]$, and let a sequence $(y_n)_{n\in\mathbb{N}}$ be defined by:

$$y_1 = x, \quad y_2 = x - \tfrac{1}{2}y_1^2, \quad \cdots, \quad y_{n+1} = x - \tfrac{1}{2}y_n^2 \ (n \geq 1).$$

Then, as

$$y_n \in [0,x], \quad y_{2n-1} \geq y_{2n+1}, \quad y_{2n} \leq y_{2n+2}, \quad y_{2n} \leq y_{2n-1}$$

for all $n \in \mathbb{N}$, we have:

$$(\star) \quad 0 \leq y_2 \leq y_4 \leq \cdots \leq y_{2n} \leq y_{2n+2} \leq$$
$$\cdots \leq y_{2n+1} \leq y_{2n-1} \leq \cdots \leq y_3 \leq y_1 = x,$$

$\therefore \lim_{n\to\infty} y_{2n-1}$ (exists,) $= \ell \in \mathbb{R}$, $\lim_{n\to\infty} y_{2n} = \ell_1 \in \mathbb{R}$.

$$\because \quad y_{2n} - y_{2n+1} = \frac{y_{2n}^2 - y_{2n-1}^2}{2}$$

$$\therefore \text{(by letting } n\to\infty) \quad \ell_1 - \ell = \frac{\ell_1^2 - \ell^2}{2}$$

$$\text{i.e.} \quad (\ell_1 - \ell)(\ell_1 + \ell - 2) = 0.$$

$\because$ $|y_k| \leq x$, and $y_3 < y_1$ when $x = 1$, we have $\ell_1 + \ell - 2 \neq 0$.

$$\therefore \quad \ell_1 = \ell.$$

This proves $\lim_{n\to\infty} y_n = \ell (\in \mathbb{R})$. As $y_{n+1} = x - \frac{1}{2}y_n^2$, we have (by letting $n\to\infty$): $\ell = x - \frac{1}{2}\ell^2$, i.e. $\ell^2 + 2\ell - 2x = 0$.

$\therefore \quad \ell = -1 + \sqrt{1+2x}$ (as $-1 - \sqrt{1+2x} < 0$).