§ Limits of functions

Consider $f: (a,b) \to \mathbb{R}$.

$\boxed{1}$ $b = \infty$,

$$\lim_{x\to\infty} f(x) = \begin{cases} l & \text{(a real number)} \\ \infty \\ -\infty \end{cases} \qquad \left(\text{similar to } \lim_{n\to\infty} a_n \right)$$

e.g.

(1) $$\lim_{x\to\infty} \frac{2x^2+1}{3x^2+7} \overset{\substack{\text{div. both num.} \\ \text{\& den. by } x^2}}{=} \lim_{x\to\infty} \frac{2+\frac{1}{x^2}}{3+\frac{7}{x^2}} \overset{\substack{\text{by intuition} \\ \text{(Thm)}}}{=} \frac{2}{3}.$$

(2) $$\lim_{x\to\infty} \frac{2x^2+3}{x+1} = \lim_{x\to\infty} \frac{2x+\frac{3}{x}}{1+\frac{1}{x}} = \infty.$$

(3) $$\lim_{x\to\infty} \frac{-2x^2+5}{x+7} = \lim_{x\to\infty} \frac{-2x+\frac{5}{x}}{1+\frac{7}{x}} = -\infty.$$

(4) $$\lim_{x\to\infty} \frac{x+7}{3x^2-8} = \lim_{x\to\infty} \frac{x/x^2+7/x^2}{3-8/x^2} = \lim_{x\to\infty} \underbrace{\frac{1/x+7/x^2}{3-8/x^2}}\ \left(\cancel{= \frac{0+0}{3-0}} \right) \overset{\text{by intuition}}{=} 0.$$

$\left[\frac{1/x+7/x^2}{3-8/x^2} \text{ is close to } \frac{0}{3} \right]$

$\boxed{2}$ $a = -\infty$,

e.g.

(5) $$\lim_{x\to-\infty} \frac{2x^2+1}{3x^2+7} = \lim_{x\to-\infty} \frac{2+\frac{1}{x^2}}{3+\frac{7}{x^2}} = \frac{2}{3}.$$

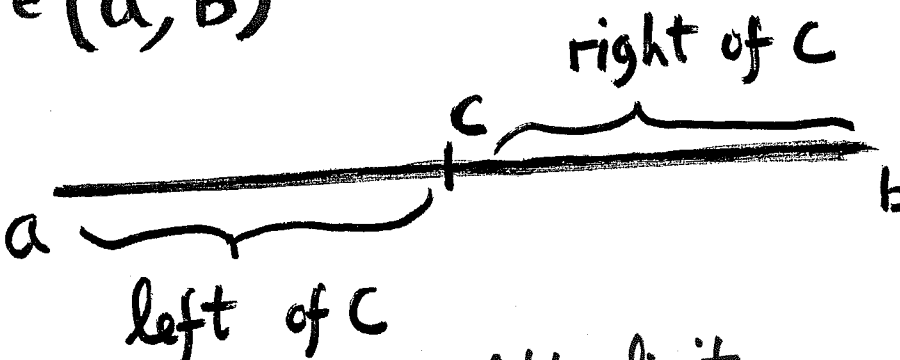
(6) $\displaystyle\lim_{x\to-\infty} \frac{2x^2+3}{x+1} = \lim_{x\to-\infty} \frac{2x+\frac{3}{x}}{1+\frac{1}{x}} = -\infty.$

(7) $\displaystyle\lim_{x\to-\infty} \frac{-2x^3+1}{3x^2+7} = \lim_{x\to-\infty} \frac{-2x+\frac{1}{x^2}}{3+\frac{7}{x^2}} = \infty.$

Skip this at present

(8) $\displaystyle\lim_{x\to-\infty} \frac{x+7}{3x^2-8} = \lim_{x\to-\infty} \frac{x/x^2+7/x^2}{3-8/x^2} = \lim_{x\to-\infty} \frac{1/x+7/x^2}{3-8/x^2} = \frac{0}{3} = 0.$

$\boxed{3}$ $c \in (a, b)$



left-limit

(*) $\displaystyle\lim_{x\to c^-} f(x) = \begin{cases} l & \text{(a real number)} \\ \infty \\ -\infty \end{cases}$ $\quad \displaystyle\lim_{x\to b^-} f(x)$

e.g.

1. $\displaystyle\lim_{x\to 0^-} \frac{1}{x} = -\infty$ $\quad$ ($\frac{1}{x}$ can be very negative with x sufficiently close to 0 and on the left of 0 (distinct from x) i.e. $x<0$).

2. $\displaystyle\lim_{x\to 0^-} \frac{1}{x^2} = \infty.$

3. $\displaystyle\lim_{x\to 2^-} x^2 = 4.$

4. Let $f(x) = \begin{cases} -x, & x > 3, \\ 0, & x = 3, \\ x^2 & x < 3. \end{cases}$

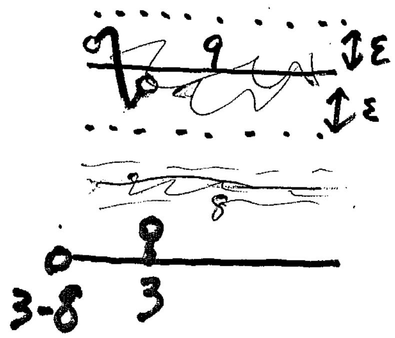
Then $\lim\limits_{x \to 3^-} f(x) = \lim\limits_{x \to 3^-} x^2 = 9$

" With $x < 3$ yet suff. close to 3,

$\overset{= x^2}{f(x)}$ is close to 9 (to an extent as you wish)."

"For any $\varepsilon > 0$, there exists $\delta > 0$ such that

$|f(x) - 9| < \varepsilon$ whenever $3 - \delta < x < 3$."



$\boxed{\text{As } \lim\limits_{x \to 3^-} f(x) \neq f(3), \text{ we say that } f \text{ has a break at } 3.}$

$$\lim_{x \to c^+} f(x) = \begin{cases} \ell \\ -\infty \\ \infty \end{cases} \quad \text{right-limit}$$

$\lim\limits_{x \to a^+} f(x)$

e.g.

5. Let f as in 4, i.e. $f(x) = \begin{cases} -x, & x > 3, \\ 0, & x = 3, \\ x^2, & x < 3. \end{cases}$

Then $\lim\limits_{x \to 3^+} f(x) = \lim\limits_{x \to 3^+} (-x) = -3$.

$\boxed{\text{As } \lim\limits_{x \to 3^+} f(x) \neq f(3), \text{ we say that } f \text{ has a break at } 3.}$

" With $x > 3$ yet suff. close to 3, $f(x)$ is close to -3."

" For any $\varepsilon > 0$, there exists $\delta > 0$ such that

$|f(x) - (-3)| < \varepsilon$ whenever $3 < x < 3 + \delta$."

(***) $$\boxed{\lim_{x\to c} f(x) = \ell \quad \text{means} \quad \lim_{x\to c^-} f(x) = \ell = \lim_{x\to c^+} f(x).}$$

e.g.

6. $\displaystyle\lim_{x\to 0} \frac{1}{x^2} = \infty$

7. $\displaystyle\lim_{x\to 2} x^2 = 4$

8. $\displaystyle\lim_{x\to 0} \frac{1}{x}$ does not exist, because $\displaystyle\lim_{x\to 0^-} \frac{1}{x} = -\infty$,

$$\lim_{x\to 0^+} \frac{1}{x} = \infty.$$

9. Let $f(x) = \begin{cases} -x, & x>3 \\ 0, & x=3, \\ x^2, & x<3. \end{cases}$

Then $\displaystyle\lim_{x\to 3} f(x)$ does not exist, because

$$\lim_{x\to 3^-} f(x) = 9 \neq -3 = \lim_{x\to 3^+} f(x).$$

$\displaystyle\lim_{x\to c} f(x)$ does not exist if

$\displaystyle\lim_{x\to c^-} f(x)$ or $\displaystyle\lim_{x\to c^+} f(x)$ does not exist

or $\displaystyle\lim_{x\to c^-} f(x) \neq \lim_{x\to c^+} f(x)$ (both limits exist but are different).

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II-13

<u>Thm 0</u>

(a) If $\lim_{x\to c^-} f(x) = \ell$ and $\lim_{x\to c^-} f(x) = \ell'$ (where
In other words
$\ell, \ell' \in \mathbb{R}\cup\{-\infty, \infty\}$), then $\ell = \ell'$. Thus if $\lim_{x\to c^-} f(x)$ exists, then it is unique.

(b) If $\lim_{x\to c^-} f(x) = \ell$, $\lim_{x\to c^-} g(x) = u$, and $f(x) \le g(x)$ for x sufficiently close to c and $x < c$, then

$$\ell \le u.$$

<u>Ex. 0</u> By geometric consideration, we have

$$\sin x < x$$

for $0 < x$ and x sufficiently close to 0. By Thm 0 (b),

$$\lim_{x\to 0^+} \sin x \le \lim_{x\to 0^+} x,$$

because both limits exist. In fact, they are both equal to 0.

Thm 1

(1) Suppose $\lim\limits_{x \to c^-} f(x)$, $\lim\limits_{x \to c^-} g(x)$ are real numbers.

Then

(i) $\lim\limits_{x \to c^-} (f(x) + g(x)) = \lim\limits_{x \to c^-} f(x) + \lim\limits_{x \to c^-} g(x)$,

(ii) $\lim\limits_{x \to c^-} (k f(x)) = k \lim\limits_{x \to c^-} f(x)$, where $k \in \mathbb{R}$,

(iii) $\lim\limits_{x \to c^-} (f(x) g(x)) = \lim\limits_{x \to c^-} f(x) \lim\limits_{x \to c^-} g(x)$

(iv) $\lim\limits_{x \to c^-} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to c^-} f(x)}{\lim\limits_{x \to c^-} g(x)}$, provided $\lim\limits_{x \to c^-} g(x) \neq 0$.

(2) $\lim\limits_{x \to c^-} \dfrac{h(x)}{f(x)} = 0$ if $\lim\limits_{x \to c^-} f(x) = -\infty$ or ∞, and if there is $k \in \mathbb{R}$ such that $|h(x)| \leq k$ for $\underset{c > x}{c \neq x}$ suff. close to c

(3) $\lim\limits_{x \to c^-} \dfrac{1}{f(x)} = \infty$ if $\lim\limits_{x \to c^-} f(x) = 0$ & $f(x) > 0$ for $\underset{c > x}{c \neq x}$ suff. close to c ;

$\lim\limits_{x \to c^-} \dfrac{1}{f(x)} = -\infty$ if $\lim\limits_{x \to c^-} f(x) = 0$ & $f(x) < 0$ for $\underset{c > x}{c \neq x}$ suff. close to c .

The tag "$x \to c^-$" can be replaced by "$x \to c^+$" or "$x \to c$" throughout
"$c > x$" — "$c < x$"
"$x \to \infty$"
"$x \to -\infty$"

Thm 2 Suppose $f(x) \leq g(x) \leq h(x)$ for x suff. close to c

and suppose $\lim_{x \to c^-} f(x) = l = \lim_{x \to c^-} h(x)$, where $l \in \mathbb{R} \cup \{-\infty, \infty\}$. Then

$$\lim_{x \to c^-} g(x) = l.$$

(Sandwich theorem) "$x \to c^-$" can be replaced by "$x \to c^+$" throughout
"$x \to c$" throughout
"$x \to \infty$"
"$x \to -\infty$"

Thm 3 Suppose $f(x)$ is increasing for x suff. close to c and $f(x) \leq k$ for a constant k and for x suff. close to c

(i.e. $f(x) \leq f(x') \leq k$ whenever $x < x'$ (both suff. close to c, $x < c$)).

Then there exists $l \in (-\infty, k]$ such that

$$\lim_{x \to c^-} f(x) = l.$$

"x is suff. close to ∞" means "x is suff. large"
"x is suff. close to $-\infty$" means "x is suff. negative."

Thm 3' Suppose there is a constant k such that

$f(x) \geq f(x') \geq k$ whenever $x < x'$ (both suff. close to c).

$x < c$ (in $\lim_{x \to c^-} f(x)$)
$x > c$ (in $\lim_{x \to c^+} f(x)$)

Then there exists $l \in [k, \infty)$ such that

$$\lim_{x \to c^-} f(x) = l.$$

"$x \to c^-$" can be replaced by "$x \to c^+$" throughout
"$x \to c$" throughout
"$x \to \infty$", "$x \to -\infty$"

More Examples

10. $$\lim_{x\to -3} \frac{x^2-9}{x+3} = \lim_{x\to -3} (x-3) = -6.$$

11. $$\lim_{x\to 0} \frac{\sqrt{x+9}-3}{x} = \lim_{x\to 0} \frac{(\sqrt{x+9})^2 - 3^2}{x(\sqrt{x+9}+3)}$$

$$= \lim_{x\to 0} \frac{x}{x(\sqrt{x+9}+3)} = \lim_{x\to 0} \frac{1}{\sqrt{x+9}+3} = \frac{1}{\sqrt{9}+3}$$

$$= \frac{1}{6}.$$

12. $$\lim_{x\to 0} \frac{(1+x)^{2/3}-1}{x} = \lim_{x\to 0} \frac{\{[(1+x)^{2/3}]-1\}\{[(1+x)^{2/3}]^2+[(1+x)^{2/3}]+1\}}{x\{[(1+x)^2]^{2/3}+[(1+x)^2]^{1/3}+1\}}$$

$$= \lim_{x\to 0} \frac{(1+x)^2-1}{x\{[(1+x)^2]^{2/3}+[(1+x)^2]^{1/3}+1\}}$$

$$\left\{\begin{array}{l}\text{recall } (b-1)(b^2+b+1) \\ \qquad = b^3-1, \\ \text{with } b=(1+x)^{2/3}.\end{array}\right\}$$

$$= \lim_{x\to 0} \frac{(2+x)\cancel{x}}{\cancel{x}(1+x)^{4/3} + (1+x)^{2/3}+1}$$

$$= \frac{\lim_{x\to 0}(2+x)}{\lim_{x\to 0}\left[(1+x)^{4/3}+(1+x)^{2/3}+1\right]} = \frac{2}{3}.$$

Ex (I skipped this page in the class, but you can study it yourself — except perhaps (6).)

(13) $\lim\limits_{x\to c}(x+7) = c+7.$

(14) $\lim\limits_{x\to c} x^2 = \left(\lim\limits_{x\to c} x\right)\left(\lim\limits_{x\to c} x\right) = c^2.$

(15) $\lim\limits_{x\to c}(7x^3+2x-8) = 7\left(\lim\limits_{x\to c} x\right)^3 + 2\left(\lim\limits_{x\to c} x\right) - 8$

$= 7c^3 + 2c - 8.$

(16)

$$\lim_{x\to 0}\frac{3x^2-7}{4x+5} = \frac{\lim\limits_{x\to 0}(3x^2-7)}{\lim\limits_{x\to 0}(4x+5)}$$

$= \frac{-7}{5}.$

(17)

$$\lim_{x\to 1}\frac{x^2-1}{x-1} = \lim_{x\to 1}\frac{(x-1)(x+1)}{x-1} = \lim_{x\to 1}(x+1)$$

$= 2.$

(18)

Given $\lim\limits_{x\to\infty}(3x+1-ax) = 1$, find a.

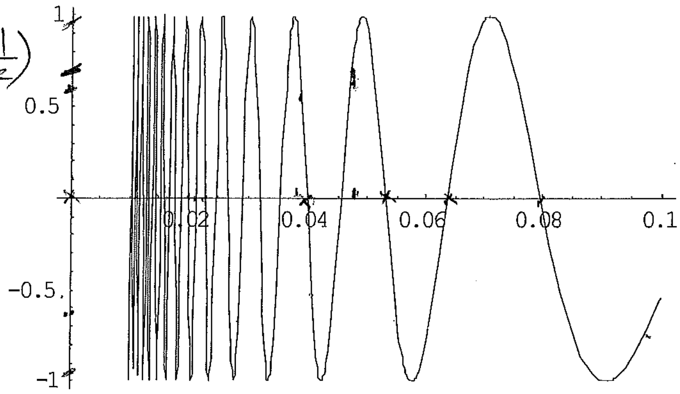
$\because \lim\limits_{x\to\infty}(3x+1-ax) = \lim\limits_{x\to\infty}\left[(3-a)x+1\right]$ cannot be a (real) number unless $3-a=0$, $\therefore$ a must be 3. On the other hand, if $a=3$, $\lim\limits_{x\to\infty}(3x+1-ax) = 1$. II-6

$\therefore\ a = 3.$

Examples

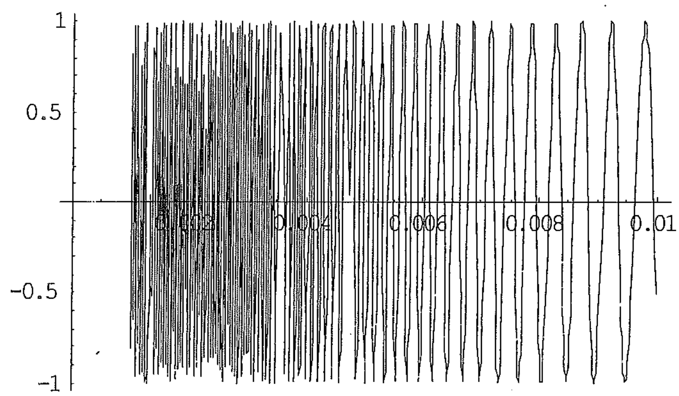
(19)

$\sin\left(\frac{1}{x}\right)$



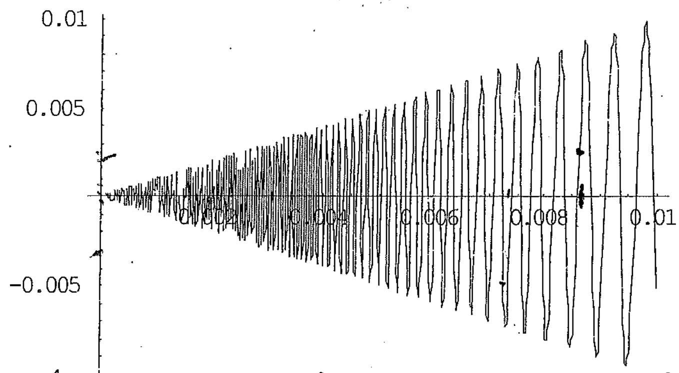
Graph of Sin (1/x)

$\lim_{x \to 0^+} \sin\left(\frac{1}{x}\right)$ does not exist



Example (20)

Graph of x Sin (1/x)



$\lim_{x \to 0} x \sin \frac{1}{x} = \underline{0}$

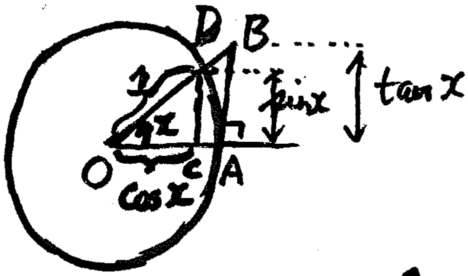
The above example (20) is an example of the following situation

$\boxed{4}$ $c \in (a,b)$, $\quad \mathrm{Dom}(f) = (a,b) \setminus \{c\}$

Ex. 21 $\displaystyle\lim_{x\to 0}\frac{\sin x}{x} = 1, \quad \lim_{x\to 0}\cos x = 1.$

you may skip this

Pf. By the diagram



$$\lim_{x\to 0^+}\cos x = 1 \quad \cdots(1)$$

$$\left(\text{similarly } \lim_{x\to 0^-}\cos x = 1\right)$$

and $\displaystyle\lim_{x\to 0}\cos x = 1.$

Moreover,

area of $\triangle OCD$ < area of sector $\widehat{OAD}$ < area of $\triangle OAB$

i.e. $\frac{1}{2}(\sin x)(\cos x) < \frac{1}{2}x < \frac{1}{2}\tan x$

$$\therefore \quad \frac{1}{\cos x} > \frac{\sin x}{x} > \cos x \quad \cdots(2)$$

By (1), (2) & Sandwich Thm, $\displaystyle\lim_{x\to 0^+}\frac{\sin x}{x} = 1$. Since $\frac{\sin x}{x}$ is even,

$$\lim_{x\to 0}\frac{\sin x}{x} = 1.$$

Similarly we have:

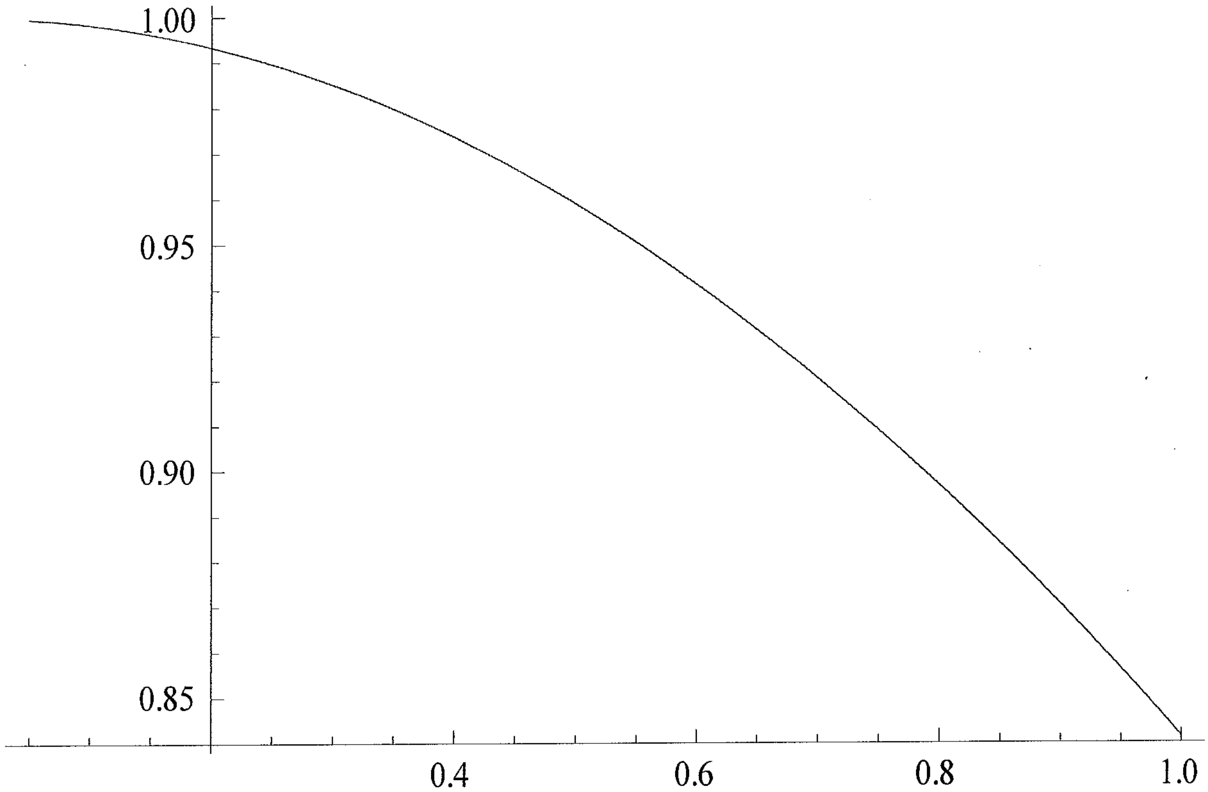
$$\lim_{x\to 0}\sin x = 0, \quad \lim_{x\to \pi/2}\sin x = 1, \quad \lim_{x\to -\pi/2}\sin x = -1$$

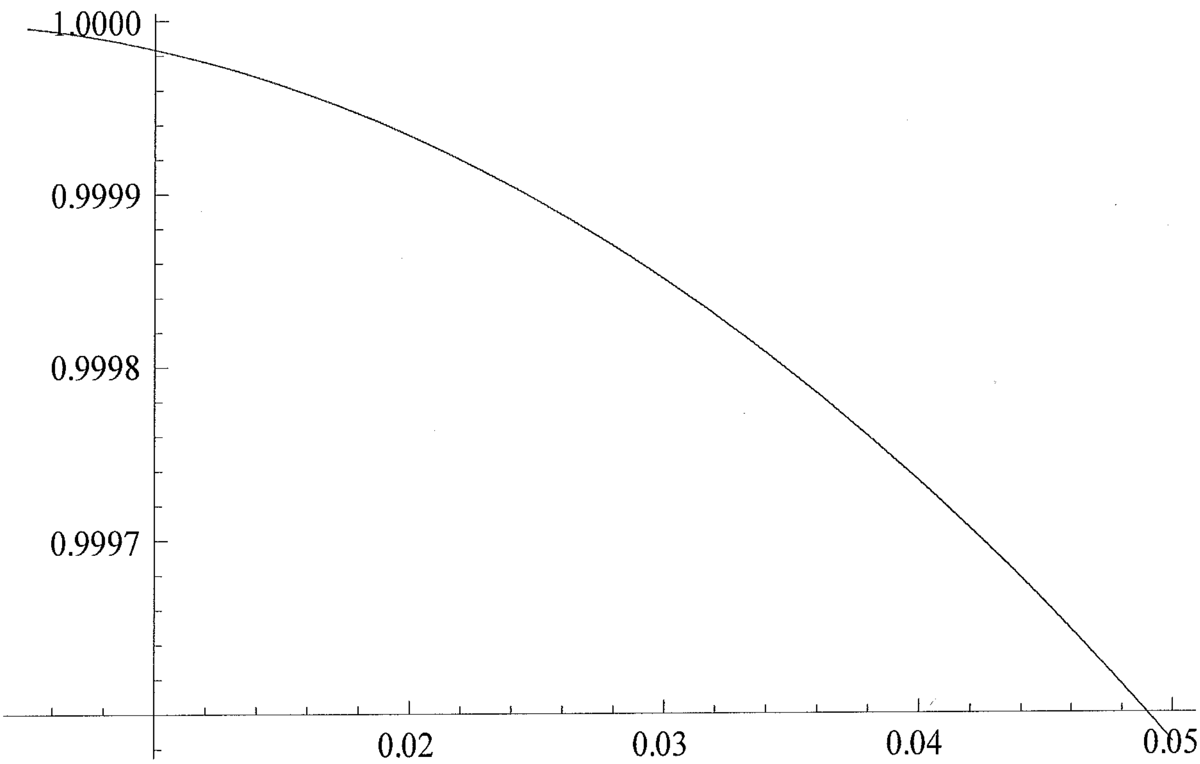
III-7

III-1

$$\lim_{x\to \pi/2}\cos x = 0 = \lim_{x\to -\pi/2}\cos x, \text{ etc.}$$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

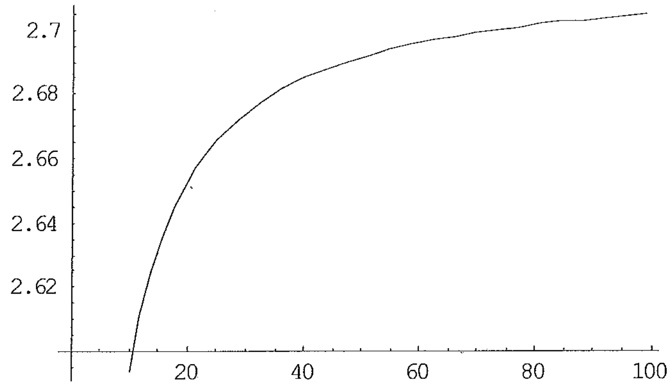




Skip this at present

$\lim_{x\to\infty}(1+1/x)^x$,

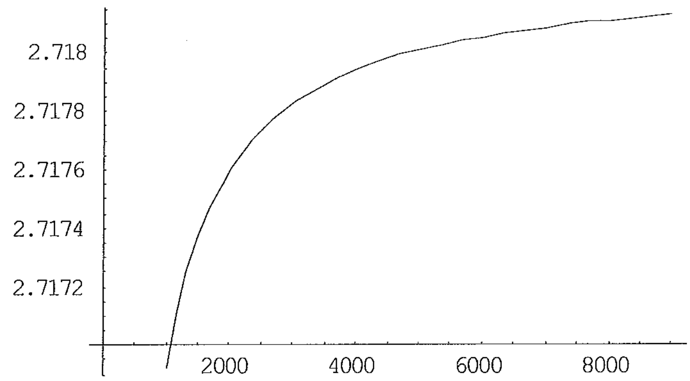
Ex.22 The number e as the limit ~~of the sequence (1+1/n)^n, n in N~~.



$(1+\frac{1}{x})^x$

$(1.05)^{20} = \cdot\cdot$

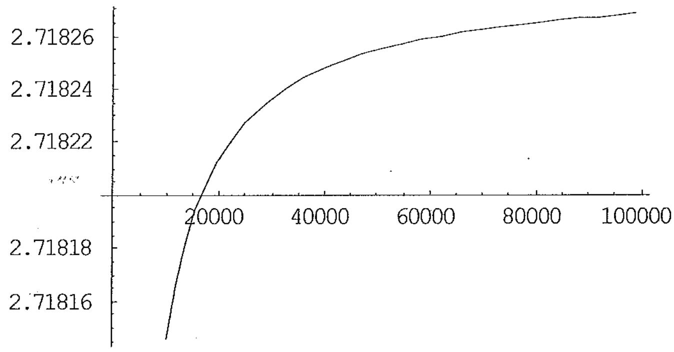
$(1+\frac{1}{40})^{40} = \cdots$

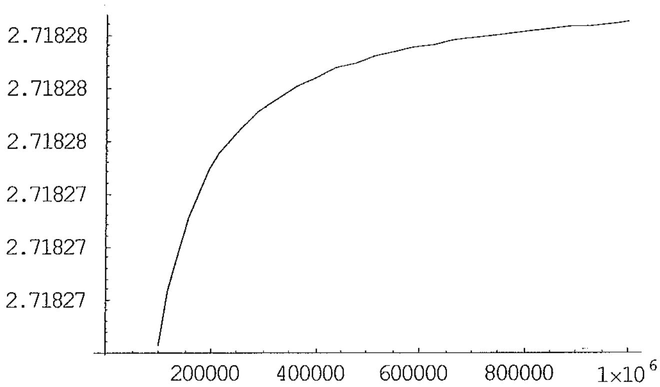


$\lim_{x\to\infty}(1+\frac{1}{x})^x = l$

$(1+\frac{1}{\sqrt{2}})^{\sqrt{2}}$

$(1.7..)^{\sqrt{2}}$





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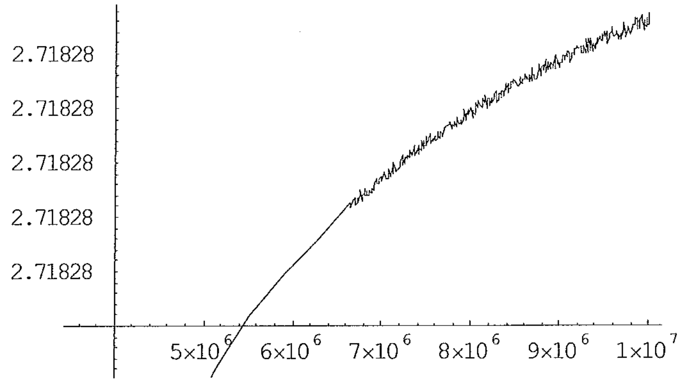


The diagram indicates the terms of the sequence are getting bigger and bigger, but they are all 2.71828 (as rounded off at the fifth decimal place) when n is large enough.

controlled by computer accuracy level.

$$e = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x.$$

The natural number

Ex. $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$.

you may skip this

We already know that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$.

Suppose $n_x \le x < n_x+1$ where n_x is a positive integer (the integral part of x) (e.g. for $x=1987.56$, $n_x=1987$ etc.)

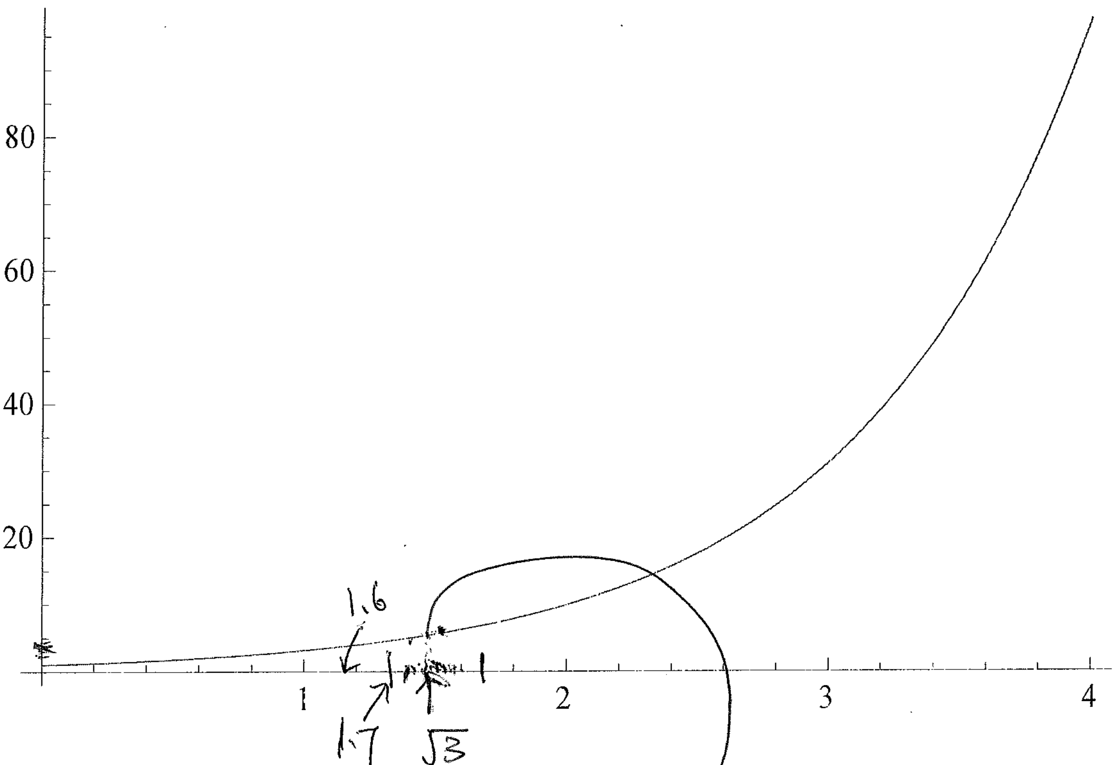
Then $\frac{1}{n_x+1} < \frac{1}{x} \le \frac{1}{n_x}$, hence

$$\underbrace{\left(1+\frac{1}{n_x+1}\right)^{-1}}_{\to 1}\underbrace{\left(1+\frac{1}{n_x+1}\right)^{n_x+1}}_{\to e} < \left(1+\frac{1}{x}\right)^x \le \underbrace{\left(1+\frac{1}{n_x}\right)^{n_x}}_{\to e}\cdot\underbrace{\left(1+\frac{1}{n_x}\right)}_{\to 1}$$

$\therefore \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$.

A Remark on Exponential functions

The graph of Pi^x π^x



What's $\pi^{\sqrt{3}}$?

$\pi^{\sqrt{3}}$ denote

positive fractions a_n, such that $\lim_{n\to\infty} a_n = \sqrt{3}$

$\lim_{n\to\infty} \pi^{a_n}$ exists as a finite no.

i.e. $\left(\pi^{a_n}\right)_{n\in\mathbb{N}}$ converges to a number, which does not depend on the choice of (a_n).

positive

Given a real b, another real no. a,

positive not a fraction

we define

$$b^{a} \overset{\text{def}}{=} \lim_{n\to\infty}\left(b^{a_n}\right)$$

where $\lim\limits_{n\to\infty} a_n = a$ and each a_n is a fraction.

It can be proved that the $\lim\limits_{n\to\infty}\left(b^{a_n}\right)$ above does not depend on the choice of (a_n).

IV-3

~~II-10.1~~

~~III-11.1~~

Ex.23. $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e = \lim\limits_{x\to-\infty}\left(1+\frac{1}{x}\right)^x$,

$\lim\limits_{y\to\infty}\left(1+\frac{x}{y}\right)^y = e^x$ for all $x\in\mathbb{R}$,

You may skip this $\lim\limits_{x\to 0}(1+x)^{1/x} = e$.

Soln We accept $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$ (as definition) and note that $e = 2.718\cdots$. Now, for $x<0$, let $y=-x$. Then

$$\left(1+\frac{1}{x}\right)^x = \left(1-\frac{1}{y}\right)^{-y} = \left(\frac{y}{y-1}\right)^y$$

$$= \left(1+\frac{1}{y-1}\right)^{y-1}\cdot\left(1+\frac{1}{y-1}\right),$$

hence

$$\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = \lim_{y\to\infty}\left(1+\frac{1}{y-1}\right)^{y-1}\cdot\lim_{y\to\infty}\left(1+\frac{1}{y-1}\right)$$

$$= e\cdot 1 = e.$$

The other formulas can be obtained similarly.

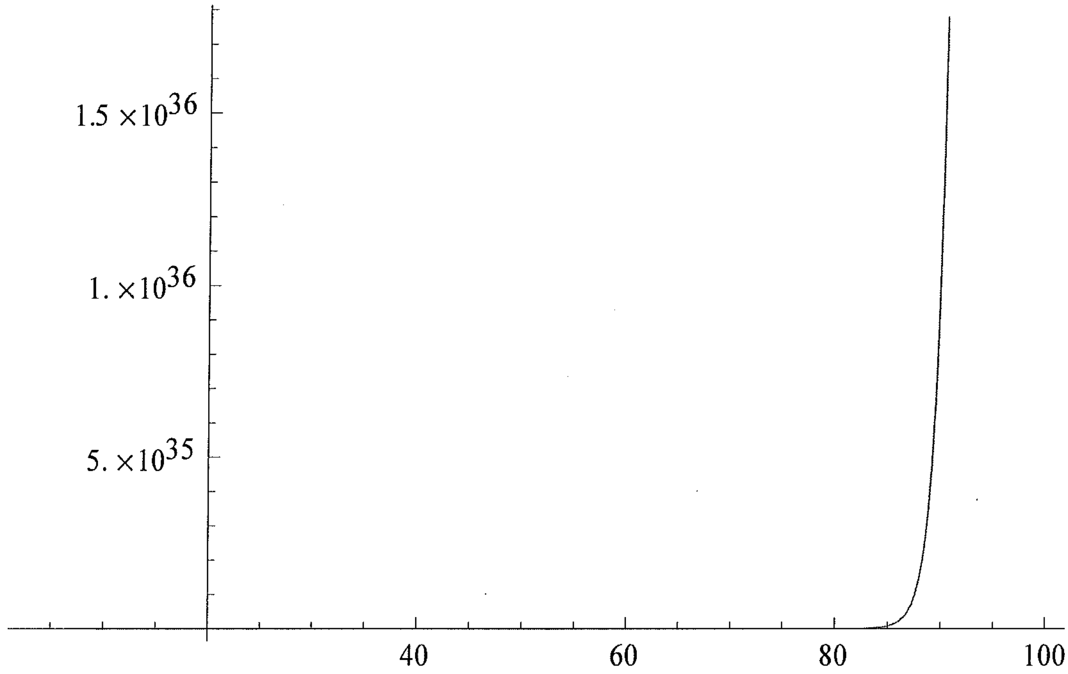
IV-5

3/4 hr 30-45

~~IV-6~~

Ex. 24

$$\lim_{x \to \infty} \left(1+\frac{1}{x}\right)^{x^2} = \infty$$

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Plot[(1+1/x)^(x^2), {x, 3, 100}]
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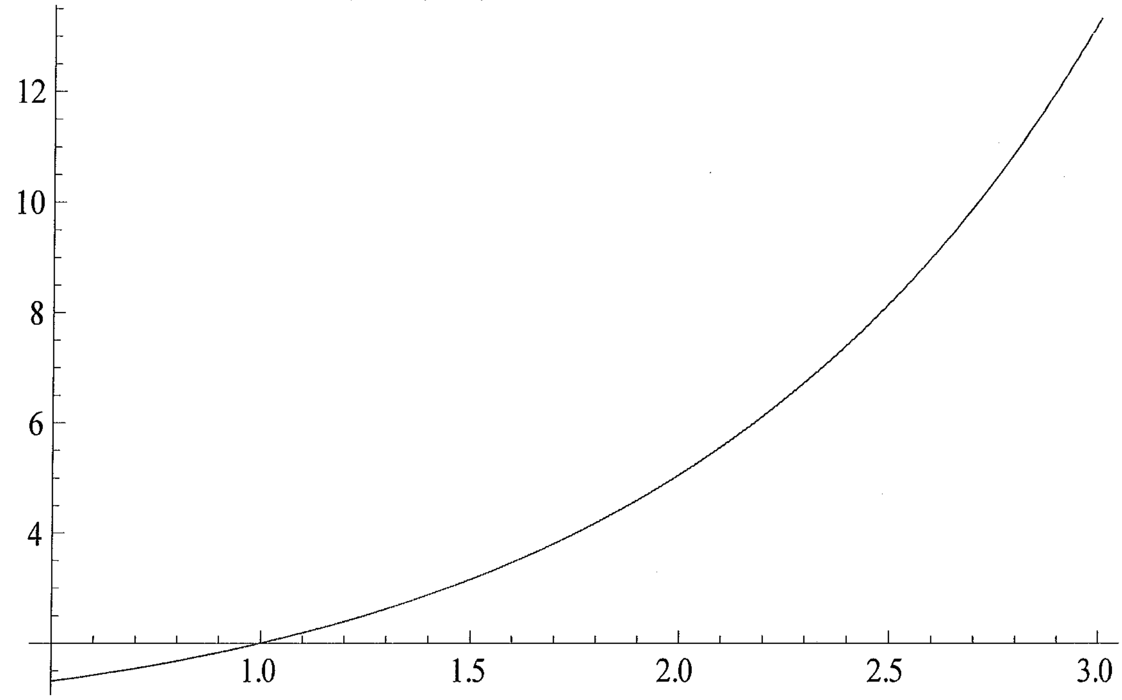


```
Plot[(1+1/x)^(x^2), {x, 0.5,3}]
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An alternative:

Because $\lim_{x \to \infty} \left(1+\frac{1}{x}\right)^{x} = e \approx 2.71$, we have: $\left(1+\frac{1}{x}\right)^{x} > 2$ when x is very large. Therefore $\left(1+\frac{1}{x}\right)^{x^2} \overset{\downarrow}{=} \left[\left(1+\frac{1}{x}\right)^{x}\right]^{x} > 2^x$ is very large, when x is very large. Thus $\lim_{x \to \infty} \left(1+\frac{1}{x}\right)^{x^2} = \infty$.