Why is the function $x \mapsto \frac{e^x - e^{-x}}{2}$ labled

$\underline{\sin}\underline{h}(x)$?

coming from $\underline{h}$yperbolic

why?

?

There is the famous Euler formula:

$$e^{ix} = \cos(x) + i\sin(x); \qquad \cdots(1)$$

(replacing x by $-x$)

$$e^{-ix} = \cos(-x) + i\sin(-x)$$

i.e. $$e^{-ix} = \cos(x) - i\sin(x) \qquad \cdots(2)$$

(1)-(2): $$e^{ix} - e^{-ix} = 2i\sin(x)$$

Formally substituting x with $-iy$,

$$e^{y} - e^{-y} = 2i\sin(-iy)$$

i.e. $$\frac{e^{y} - e^{-y}}{2} = i\,\underline{\sin}(-iy)$$

here comes the $\underline{\sin}$.

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This connection answers our curiosity, but not important for our course.

Basic Properties of Real Numbers

1. For each $a \in \mathbb{R}$, there is one and only one possibilities among:

$$a < 0, \quad a = 0, \quad a > 0.$$

2. (We say $a > b$ if and only if $a - b > 0$.)
   If $a, b > 0$, then $a + b > 0$ and $ab > 0$.

3. (For $a \in \mathbb{R}$, $|a| \overset{\text{def}}{=} \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$,
   called the <u>absolute value</u> of a.)

   For all $a, b \in \mathbb{R}$, we have

   $|a + b| \leq |a| + |b|$ (triangle inequality),

   $|ab| = |a|\,|b|$

   $|a| \geq 0$ and $|a| = 0$ iff $a = 0$.

4. If $a > b$ and $c > 0$, then $ac > bc$;
   if $a > b$ and $c < 0$, then $ac < bc$.

5. For $a \in \mathbb{R}$ and $b \in \mathbb{R} \setminus \{0\}$, we have:

$$\frac{a}{b} \geqslant 0 \quad \text{iff} \quad \text{either } \text{“}a \geqslant 0,\ b > 0\text{”} \text{ or } \text{“}a \leqslant 0,\ b < 0\text{”}.$$

6. For $b \geqslant 0$:

$$|a| \leq b \quad \text{iff} \quad -b \leq a \leq b\ ;$$

$$|a| < b \quad \text{iff} \quad -b < a < b\ .$$

Examples

1. Let $f(x) = 1/\sqrt[3]{x}$, $g(x) = \underline{x^2 - 9x}$.

Then the natural domain of f is $\underline{\mathbb{R}\setminus\{0\}}$;

the domain of g is $\mathbb{R}$;

the domain of $g\circ f$ is

$$\{x \in \mathrm{dom}(f) : f(x) \in \mathrm{dom}(g)\}$$

$$= \{x \in \mathbb{R}\setminus\{0\} : \frac{1}{\sqrt[3]{x}} \in \mathbb{R}\}$$

$$= \mathbb{R}\setminus\{0\} \quad (= (-\infty, 0) \cup (0, \infty));$$

the domain of $f\circ g$ is:

$$\{x \in \mathrm{dom}(g) : g(x) \in \mathrm{dom}(f)\}$$

$$= \{x \in \mathbb{R} : x^2 - 9x \neq 0\}$$

$$= \{x \in \mathbb{R} : x \neq 0 \text{ and } x \neq 9\}$$

$$= \mathbb{R}\setminus\{0, 9\}$$

and $f\circ g$ / $\mathbb{R}\setminus\{0,9\}$ with

$$f\circ g\,(x) = f(g(x)) = 1/\sqrt[3]{g(x)}$$

$$= 1/\sqrt[3]{x^2 - 9x}, \quad x \in \mathbb{R}\setminus\{0,9\}.$$

2. The natural domain of the function

$$g(x) = \sqrt{\frac{x}{x^2-1}}$$

is: $\left\{x \in \mathbb{R} : \frac{x}{x^2-1} \geq 0\right\}$

$= \{x \in \mathbb{R} : x \geq 0,\ x^2-1 > 0\} \cup \{x \in \mathbb{R} : x \leq 0,\ x^2-1 < 0\}$

$= \{x \in \mathbb{R} : x > 1\} \cup \{x \in \mathbb{R} : x \leq 0,\ x > -1\}$

$= (-1, 0] \cup (1, \infty)$.

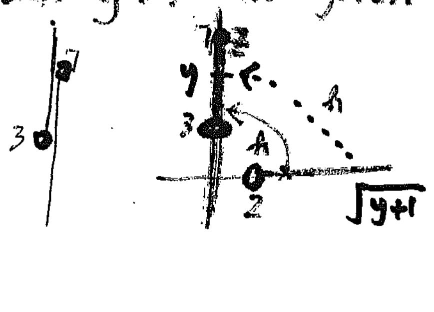
3. The natural domain of

$$h(x) = \begin{cases} x^2-1, & x > 2 \\ 3x+1, & x < 2 \end{cases}$$

is $\{x \in \mathbb{R} : x > 2 \text{ or } x < 2\} = \mathbb{R} \setminus \{2\}$.

The range of h is $\mathbb{R}$, because

(i) $h(x) > 3$ if $x > 2$, (ii) each $y > 3$ is given by $h(\sqrt{y+1})$ where $\sqrt{y+1} > 2$,



(iii) $h(x) < 7$ if $x < 2$,

(iv) each $z < 7$ is given by $h\left(\frac{z-1}{3}\right)$ where $\frac{z-1}{3} < 2$.