§ 1. Sets

A ***set*** is a collection of objects. These objects are called ***elements*** of the set, either listed out or described by properties they share.

Example 1.
$\mathbb{N} \overset{\text{def}}{=} \{1, 2, \cdots\}$, or, $\mathbb{N} \overset{\text{def}}{=} \{x : x \text{ is a natural number}\}$; $\mathbb{Z} \overset{\text{def}}{=} \{0, \pm 1, \pm 2 \cdots\}$, or, $\mathbb{Z} \overset{\text{def}}{=} \{x : x \text{ is an integer}\}$; $\mathbb{R} \overset{\text{def}}{=} \{x : x \text{ is a real number}\}$;
$\mathbb{R}^2 \overset{\text{def}}{=} \{(x, y) : x, y \in \mathbb{R}\}$; $\mathbb{R}^3 \overset{\text{def}}{=} \{(x, y, z) : x, y, z \in \mathbb{R}\}$;
$A \overset{\text{def}}{=} \{x : x \text{ is a solution of } 3x^2 - 7 = 0\}$

We say that these objects ***belong to*** the set, and write e.g. $b \in B$; if c does not belong to a set B, write $c \notin B$.
Two sets A and B are ***equal***, i.e. $A = B$, means $a \in A \iff a \in B$.
We say: A is a ***subset*** of B, in symbol, $A \subset B$, or $B \supset A$, if $x \in B$ whenever $x \in A$. If A is not a subset of B, we write $A \not\subset B$, or $B \not\supset A$.

Example 2. $0 \notin \mathbb{N}, 2 \in \mathbb{N}, \sqrt{2} \notin \mathbb{Z}, -3 \in \mathbb{Z}$. $\mathbb{Z} \subset \mathbb{R}$, $\mathbb{Z} \not\subset \mathbb{N}$, $\{0, 1\} \not\subset \{2\}$, $\{2\} \not\subset \{0, 1\}$.

Denote by $\emptyset$ the set without any element, called the ***empty set***. Thus $\emptyset = \{x : x \neq x\}$.
We often define a subset C of a given set D by specifying the property of C's elements, e.g. $\mathbb{N} = \{x \in \mathbb{Z} : x > 0\}$; for $a, b \in \mathbb{R}$, $[a, b] \overset{\text{def}}{=} \{x \in \mathbb{R} : a \leq x \leq b\}$, $[a, b) \overset{\text{def}}{=} \{x \in \mathbb{R} : a \leq x < b\}$, and $(a, b]$, $(a, b), (-\infty, b)$, $(-\infty, b], (a, \infty), [a, \infty)$ are similarly defined.
Venn Diagrams

§ 2. Unions, Intersections, Complements

For two sets A, B, we define the ***union*** $\mathbf{A} \cup \mathbf{B}$ of A and B by:
$A \cup B \overset{\text{def}}{=} \{x : x \in A \text{ or } x \in B\}$

Example 3. For $A = \{0, 1\}$, $B = \{0, 4\}$,
$\{0, 1\} \cup \{0, 4\} = \{0, 1, 4\}$.

The ***intersection*** $A \cap B$ (of sets A, B) is:

$A \cap B \overset{\textbf{def}}{=\!=} \{x : x \in A \text{ and } x \in B\}$.

Example 4.

$(-5, 4) \cap [-3, 8) = [-3, 4)$.

We define the ***difference*** $A \setminus B$ of A less B by:

$A \setminus B \overset{\textbf{def}}{=\!=} \{x : x \in A \text{ but } x \notin B\}$.

$A \setminus B$ is also called the **complement** of B in A.

Example 5.

Let $A = \mathbb{N}$, $B = \{$ even integers $\}$; then

$A \setminus B = \{1, 3, 5, \cdots\}$,

$B \setminus A = \{0, -2, -4, \cdots\}$.

Example 6. Let $A = [-4, 3)$, $B = (-1, 7)$. Then $A \setminus B = [-4, -1]$, $B \setminus A = [3, 7)$, $A \cap B = (-1, 3)$, $A \cup B = [-4, 7)$.

§ 3. Mappings

A **mapping**, or **map**, f, consists of:

(1) a set X, called the **domain**, which is also denoted as dom(f), and

(2) a **rule of correspondence** by which each element x of X is associated with a unique element $f(x)$.

We say that f is a map from X, and write f/X. It is understood that $\{f(x) : x \in X\}$ is a set, which is called the **range of f**, denoted by ran(f). f is called a map from X (in)to Y , and we write: $f : X \to Y$, if Y is a set containing ran(f). We also write: $f : X \to Y : x \mapsto f(x)$, or, $f(x)$, $x \in X$.

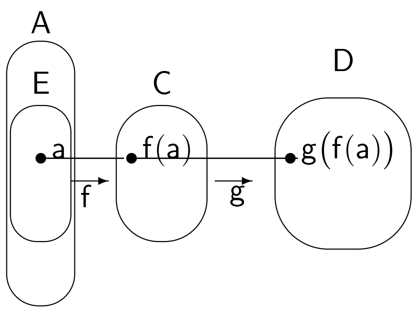
Example 7. Here is a map $f : [-1; 1] \to [-1; 2] : x \mapsto x^3$.

Two mappings f, g are **equal**, written $f = g$, if $\mathrm{dom(f)} = \mathrm{dom(g)}$, and for all $x \in \mathrm{dom(f)} = \mathrm{dom(g)}, \mathrm{f(x)} = \mathrm{g(x)}$.

Example 8. Let $A = B = \mathbb{R}$. The two mappings s, t given by:
$s(x) = \sqrt{x^2}, \;\; x \in A,$
$t(x) = |x|, \;\; x \in A,$
are equal.

Sometimes, we consider a rule of correspondence, e.g. $x \mapsto \sqrt{7x^2 - 3}$, as a mapping, with the implicit understanding that the domain is the set of all x such that the rule of correspondence (or the formula) makes sense, e.g. $\{x \in \mathbb{R} : |x| \geq \sqrt{3/7}\}$.

Let $f : A \to B, \;\; g : C \to D$. Then we define $g \circ f : E \to D$,
$(g \circ f)(a) = g\big(f(a)\big), \;\; a \in E,$
where
$E \stackrel{\mathbf{def}}{=} \{x \in dom(f) | f(x) \in dom(g)\}$. The map $g \circ f$ is called the ***composite*** of g by f.



Example 9. Let
$f : \mathbb{R} \to \mathbb{R} : x \mapsto 2x, \;\; g : \mathbb{R} \to \mathbb{R} : x \mapsto x + 7$. Then
$\big(g \circ f\big)(x) = (2x) + 7,$
$\big(f \circ g\big)(x) = 2(x + 7).$
Thus $\;g \circ f \neq f \circ g$, though they are both maps from $\mathbb{R}$ to $\mathbb{R}$.