Exercise:

1. The set of all upper triangular \( n \times n \) matrices is a subspace \( W \) of \( M_{n \times n}(\mathbb{F}) \). Find a basis for \( W \) and calculate its dimension.

2. Let \( W_1 \) and \( W_2 \) be subspaces of a finite dimensional vector space \( W \). Determine the necessary and sufficient conditions on \( W_1 \) and \( W_2 \) so that \( \dim(W_1 \cap W_2) = \dim(W_1) \).

3. Let \( V \) and \( W \) be vector spaces and \( T : V \rightarrow W \) be a linear transformation.
   (a) Prove \( T \) is one-to-one if and only if \( T \) carries linearly independent sets of \( V \) onto linearly independent subsets of \( W \).
   (b) Suppose \( T \) is one-to-one and \( S \) is a subset of \( V \). Prove \( S \) is linearly independent if and only if \( T(S) \) is linearly independent.
   (c) Suppose \( \beta = \{v_1, v_2, \ldots, v_n\} \) is a basis for \( V \), and \( T \) is one-to-one and onto. Prove \( T(\beta) = \{T(v_1), T(v_2), \ldots, T(v_n)\} \) is a basis for \( W \).

4. Let \( V \) and \( W \) be vector spaces. Let \( T : V \rightarrow W \) be a linear transformation, and \( \{w_1, w_2, \ldots, w_k\} \) be linearly independent subset of \( R(T) \). Prove that if \( S = \{v_1, v_2, \ldots, v_k\} \) is chosen so that \( T(v_i) = w_i \) for \( i = 1, 2, \ldots, k \), then \( S \) is linearly independent.
Solution:

1. Let $M_{ij}$ be the matrix whose entries are all zero except the $ij$-th entry, which is one. Let $T = \{M_{ij}|i \leq j\}$, which is a subset of $W$. We will show $T$ is a basis for $W$, so that the dimension of $W$ is $|T| = (1 + n)n/2$.

Firstly, show that $T$ generates $W$. Let $M \in W$ be an arbitrary upper triangular matrix with entries $m_{ij}$, then $m_{ij} = 0$ if $i > j$ since $M$ is an upper triangular matrix. We can see that $M = \sum_{i\leq j} m_{ij} M_{ij}$. Hence $M$ is a linear combination of $T$. Therefore, $T$ generates $W$.

It remains to show that $T$ is linearly independent. Let $A = \sum_{i\leq j} c_{ij} M_{ij} = O$. Then we can see that $A$ is upper triangular and the $ij$-th entry $a_{ij} = c_{ij}$ if $i \leq j$. Hence the equation has only trivial solution, i.e. $c_{ij} = 0$ for all $1 \leq i \leq j \leq n$. Therefore, $T$ is linearly independent, and it is a basis for $W$.

2. The necessary and sufficient condition is $W_1 \subseteq W_2$.

If $W_1 \subseteq W_2$, then $W_1 \cap W_2 = W_1$, so that $\dim(W_1 \cap W_2) = \dim(W_1)$. Hence it is a sufficient condition.

On the other hand, assume $n = \dim(W_1 \cap W_2) = \dim(W_1)$. Since $W$ is finite dimensional, we have $n$ is finite. Let $\beta$ be a basis for $W_1 \cap W_2$, then it has $n$ elements and they are linearly independent. Since $W_1 \cap W_2 \subseteq W_1$, we can conclude $\beta \subseteq W_1$, so that it is also a basis for $W_1$, which implies it generates $W_1$ and $W_1 \cap W_2 = W_1$. Therefore, $W_1 \subseteq W_2$. It is a necessary condition.

3. (a) Let $S = \{v_1, v_2, ..., v_k\}$ be arbitrary linearly independent subset of $V$, then $T(S) = \{T(v_1), T(v_2), ..., T(v_k)\}$ is a subset of $W$. Consider the equation $c_1 T(v_1) + c_2 T(v_2) + ... + c_k T(v_k) = 0$. LHS (short for left hand side) is $T(c_1 v_1 + c_2 v_2 + ... + c_k v_k)$ since $T$ is linear.

First, assume $T$ is one-to-one, then $T(x) = 0$ implies $x = 0$. Hence $T(c_1 v_1 + c_2 v_2 + ... + c_k v_k) = c_1 T(v_1) + c_2 T(v_2) + ... + c_k T(v_k) = 0$ implies $c_1 v_1 + c_2 v_2 + ... + c_k v_k = 0$. Then $c_1 = c_2 = ... = c_k = 0$ since $S$ is linearly independent. Therefore, $T(S)$ is linearly independent.

On the other hand, assume $T(S)$ is linearly independent for each linearly independent subset $S$. Prove by contradiction. Assume $T$ is not one-to-one. Then there exist some vectors $x_1, x_2$ such that $T(x_1) = T(x_2)$ and $x_1 \neq x_2$. Let $x = x_1 - x_2$, then $x \neq 0$, and $T(x) = T(x_1 - x_2) = T(x_1) - T(x_2) = 0$. Let $S = \{x\}$, then $S$ is linearly independent since $x \neq 0$. However, $T(S) = \{0\}$ is not linearly independent. Contradiction. Therefore, $T$ is one-to-one.
(b) Suppose \( S \) is linearly independent. Since \( T \) is one-to-one, from part (a) we can see that \( T(S) \) is linearly independent.

On the other hand, suppose \( T(S) \) is linearly independent. Assume \( S \) is not linearly independent. Let \( S = \{v_1, v_2, \ldots, v_k\} \), then there exist \( c_1, c_2, \ldots, c_k \) such that they are not all zero and
\[
c_1 v_1 + c_2 v_2 + \ldots + c_k v_k = 0.
\]
We get
\[
c_1 T(v_1) + c_2 T(v_2) + \ldots + c_k T(v_k) = T(c_1 v_1 + c_2 v_2 + \ldots + c_k v_k) = T(0) = 0.
\]
Contradict to the condition that \( T(S) \) is linearly independent. Hence if \( T(S) \) is linearly independent, we have \( S \) is linearly independent.

(c) First, from \( \beta \) is a basis for \( V \), we have \( \beta \) is linearly independent. And \( T \) is one-to-one, we can conclude that \( T(\beta) \) is linearly independent from part (a).

It remains to show that \( T(\beta) \) generates \( W \). For arbitrary vector \( w \) in \( W \), we can find \( v \) in \( V \) such that \( T(v) = w \) because \( T \) is onto. Since \( \beta \) is a basis for \( V \), there exist \( c_1, c_2, \ldots, c_n \) such that
\[
v = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n.
\]
Then
\[
w = T(v) = T(c_1 v_1 + c_2 v_2 + \ldots + c_n v_n) = c_1 T(v_1) + c_2 T(v_2) + \ldots + c_k T(v_k).
\]
Since \( w \) is arbitrary vector in \( W \), we conclude that \( W \) is generated by \( T(\beta) \).

Therefore, \( T(\beta) \) is a basis for \( W \).

4. Consider the equation \( c_1 v_1 + c_2 v_2 + \ldots + c_k v_k = 0 \).
If the equation above holds, we have \( 0 = T(0) = T(c_1 v_1 + c_2 v_2 + \ldots + c_k v_k) = c_1 T(v_1) + c_2 T(v_2) + \ldots + c_k T(v_k) = c_1 w_1 + c_2 w_2 + \ldots + c_k w_k \).
Since \( w_1, w_2, \ldots, w_k \) are linearly independent, we can conclude that \( c_1 = c_2 = \ldots = c_k = 0 \).
Therefore, the equation \( c_1 v_1 + c_2 v_2 + \ldots + c_k v_k = 0 \) has only trivial solution, which implies that \( S \) is linearly independent.