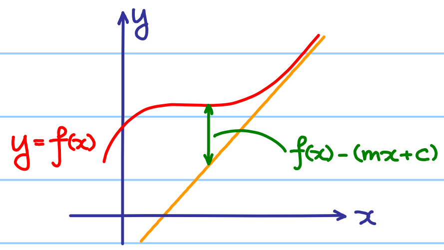
Oblique asymptote

If $y=mx+c$ is a straight such that $\lim_{x\to+\infty} f(x)-(mx+c)=0$, then the straight line is called an oblique asymptote of $f(x)$.

(Similar definition can be made for $-\infty$)



the distance tends to 0 as $x\to+\infty$

Suppose $y=mx+c$ is an oblique asymptote.

i.e. $\lim_{x\to+\infty} f(x)-mx-c=0$

Note : $\lim_{x\to+\infty}\frac{c}{x}=0$

$$\lim_{x\to+\infty}\frac{f(x)}{x}-m-\frac{c}{x}=\lim_{x\to+\infty}[f(x)-mx-c]\cdot\frac{1}{x}=0\cdot 0=0$$

Then $\lim_{x\to+\infty}\frac{f(x)}{x}=\lim_{x\to+\infty}\left(\frac{f(x)}{x}-m-\frac{c}{x}\right)+\left(m+\frac{c}{x}\right)=0+m=m$

i.e. if an oblique asymptote exists, the slope $m=\lim_{x\to+\infty}\frac{f(x)}{x}$ —— (*)

Finding oblique asymptote

Compute $\lim_{x\to+\infty}\frac{f(x)}{x}$, define it to be m if exists.

Then, compute $\lim_{x\to+\infty} f(x)-mx$, define it to be c if exists.

If both limits exist, $y=mx+c$ is an oblique asymptote.

Remark :

1) If $m=0$, it is just a horizontal asymptote, and in this case, $c=\lim_{x\to+\infty} f(x)$.

2) Even $\lim_{x\to+\infty}\frac{f(x)}{x}$ exists, we define it to be m.

$\lim_{x\to+\infty} f(x)-mx$ may NOT exist ! Any example ? (Think : $f(x)=\sqrt{x}$)

i.e. Converse of (*) is NOT true !

e.g. Let $f(x) = \dfrac{x|x-2|}{x-1}$, $x \neq 1$.

$$f(x) = \begin{cases} \dfrac{x(x-2)}{x-1} & \text{if } x \geq 2 \\ -\dfrac{x(x-2)}{x-1} & \text{if } x < 2 \text{ and } x \neq 1 \end{cases}$$

Ex: (a) Show that f is NOT differentiable at $x = 2$.

Hint: Show that $\lim\limits_{\Delta x \to 0} \dfrac{f(2+\Delta x) - f(2)}{\Delta x}$ does NOT exist.

(b) $$f'(x) = \begin{cases} \dfrac{x^2-2x+2}{(x-1)^2} & \text{if } x > 2 \\ -\dfrac{x^2-2x+2}{(x-1)^2} & \text{if } x < 2 \text{ and } x \neq 1 \end{cases}$$

Solve $f'(x) > 0$ and $f'(x) < 0$

Ans: $f'(x) > 0$ when $x > 2$

$f'(x) < 0$ when $x < 2$ and $x \neq 1$

min = $(2, 0)$

(c) $$f''(x) = \begin{cases} \dfrac{-2}{(x-1)^3} & \text{if } x > 2 \\ \dfrac{2}{(x-1)^3} & \text{if } x < 2 \text{ and } x \neq 1 \end{cases}$$

Solve $f''(x) > 0$ and $f''(x) < 0$

Ans: $f''(x) > 0$ when $1 < x < 2$

$f''(x) < 0$ when $x > 2$ or $x < 1$

point of inflection = $(2, 0)$

(d) vertical asymptote : $x = 1$

oblique asymptote :

$$\lim_{x \to +\infty} \frac{f(x)}{x} = \lim_{x \to +\infty} \frac{x(x-2)}{x(x-1)} = 1 \qquad \therefore m = 1$$

$$\lim_{x \to +\infty} f(x) - mx = \lim_{x \to +\infty} \frac{x(x-2)}{x-1} - x = \lim_{x \to +\infty} \frac{-x}{x-1} = -1 \qquad \therefore c = -1$$

oblique asymptote : $y = x - 1$

Ex: How about $-\infty$? Ans: $y = -x + 1$

(e) x-intercept: Solve $f(x) = 0$

$$\frac{x|x-2|}{x-1} = 0$$

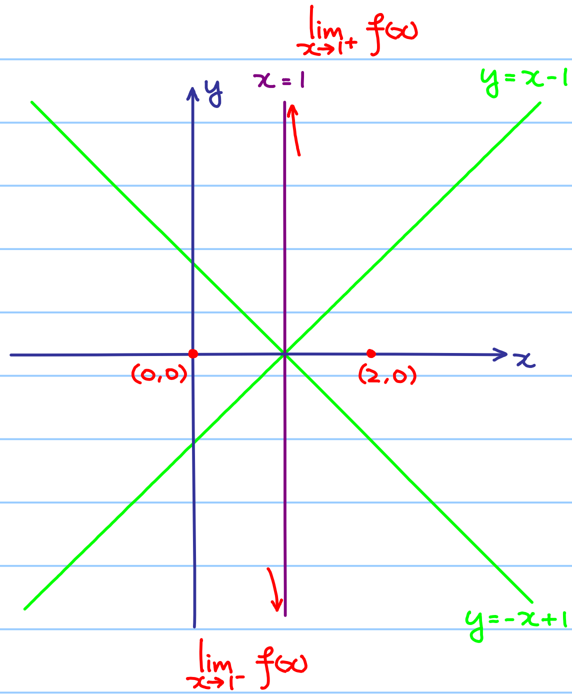
$$x = 0 \text{ or } 2$$

y-intercept: $f(0) = 0$.

(f) Sketch $y = f(x)$.

Step 1 : draw asymptotes

Step 2 : put down x-intercepts and y-intercept

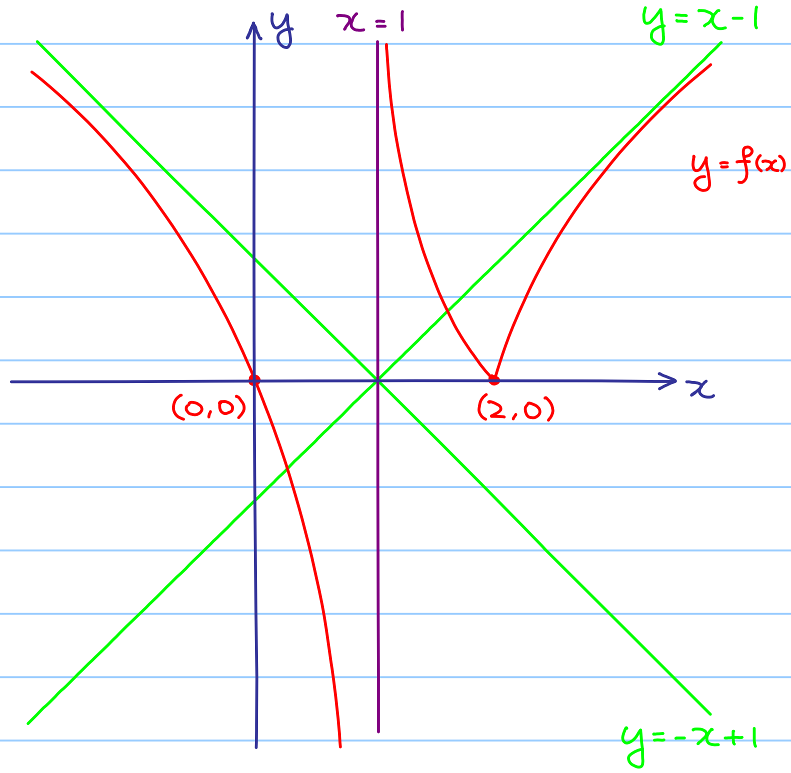


Step 3 :

$$\lim_{x\to1^-} f(x) = \lim_{x\to1^-} -\frac{x(x-2)}{x-1} = -\infty$$

$$\lim_{x\to1^+} f(x) = \lim_{x\to1^+} -\frac{x(x-2)}{x-1} = +\infty$$

Step 4 : Use the information $f'(x)$ and $f''(x)$



		1		2	
$f'(x)$	$-$	NOT defined	$-$	NOT defined	$+$
$\Downarrow$ $f(x)$	dec.		dec.		inc.

		1		2	
$f''(x)$	$-$	NOT defined	$+$	NOT defined	$-$
$\Downarrow$ $f(x)$	convex		concave		convex

Curve Sketching :

Goal : Given a function $f(x)$, sketch the graph of $y = f(x)$.

(Capturing main features)

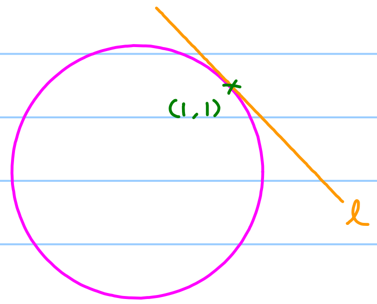
- x-intercept — solve $f(x) = 0$
- y-intercept — y-intercept $= f(0)$
- increasing / decreasing — solve $f'(x) > 0$ / $f'(x) < 0$
  saddle point / max. / min. — change of sign of $f'(x)$?
- concave / convex — solve $f''(x) > 0$ / $f''(x) < 0$
  point of inflection — change of sign of $f''(x)$?
- vertical asymptote — any $x = a$ with $\lim_{x \to a^+} f(x) = \pm\infty$ or $\lim_{x \to a^-} f(x) = \pm\infty$
- horizontal asymptote — $m = \lim_{x \to +\infty} \frac{f(x)}{x}$
- oblique asymptote — $c = \lim_{x \to +\infty} f(x) - mx$

Implicit Differentiation

e.g. $x^2+y^2=2$ —— $\mathcal{C}$

Locus of $\mathcal{C}$ is a circle centered at $(0,0)$ with radius $\sqrt{2}$.

Check: $(1,1)$ is a point lying on the circle.



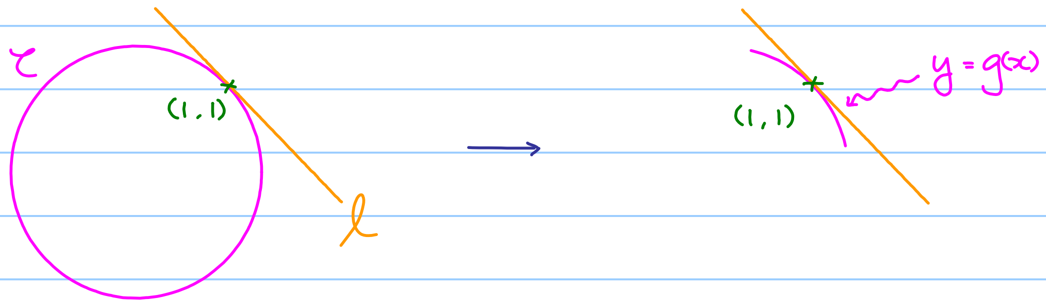
We want to find the equation of the tangent line ℓ

(i.e. need to know the slope of ℓ)

Note: $x^2+y^2=2$ is NOT a function.

Question: How to find $\frac{dy}{dx}$? (and, actually, is it defined?)

Answer: Yes, roughly speaking,



The small segment of $\mathcal{C}$ containing $(1,1)$ can be regarded as the graph of some function $y=g(x)$. (In fact, $y=\sqrt{2-x^2}$ in this case.)

How to find? Do it as usual!

eg. $x^2+y^2=2$

differentiate both sides with respect to x.

$$2x+\frac{d}{dx}y^2=0$$

$$2x+2y\frac{dy}{dx}=0 \qquad \text{(Applying chain rule)}$$

$$\frac{dy}{dx}=-\frac{x}{y}$$

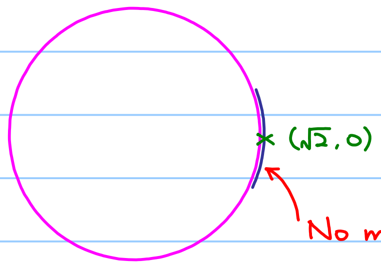
$\therefore \frac{dy}{dx}=-1$ when $(x,y)=(1,1)$.

We denote it by $\left.\frac{dy}{dx}\right|_{(x,y)=(1,1)}=-1$

Remark :

$\frac{dy}{dx}$ is defined at a point of a curve only if a small arc containing the point can be regarded as the graph of some function $y=g(x)$.

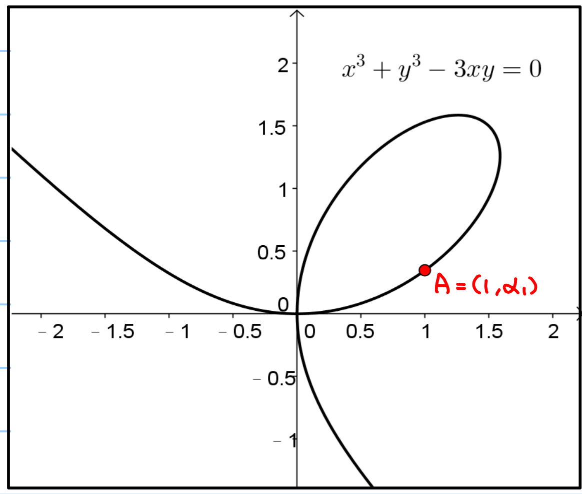
$\therefore$ $\frac{dy}{dx}$ is NOT defined when $(x,y)=(\pm\sqrt{2},0)$.



No matter how small the arc is,

it cannot be realized as graph of some function $y=g(x)$.

e.g. $x^3+y^3-3xy=0$ —— $\mathcal{C}$



$$3x^2+3y^2\frac{dy}{dx}-3y-3x\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{y-x^2}{y^2-x}$$

If we want to find the slope of the tangent line at A.

putting $x=1$ into $\mathcal{C}$.

$y^3-3y+1=0$

NOT easy to solve !

FACT : The above equation has three roots, two roots α_1,α_2 are positive $(\alpha_1<\alpha_2)$

one root is negative.

$A=(1,\alpha_1)$ and what we need is $\left.\frac{dy}{dx}\right|_{(x,y)=(1,\alpha_1)}$

Applications :

e.g. Differentiation of Logarithmic Function

Let $y = \ln x$, $x > 0$. Then $e^y = x$,

differentiate both sides with respect to x .

$$e^y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

$\therefore \frac{d}{dx} \ln x = \frac{1}{x}$ for $x > 0$.

Ex : By rewriting $\log_a x = \frac{\ln x}{\ln a}$, show that $\frac{d}{dx} \log_a x = \frac{1}{x \ln a}$.

e.g. Let $y = \ln|x|$, $x \neq 0$. Find $\frac{dy}{dx}$.

We can rewrite $y = \begin{cases} \ln x & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}$

For $x > 0$, we have just shown that $\frac{dy}{dx} = \frac{1}{x}$

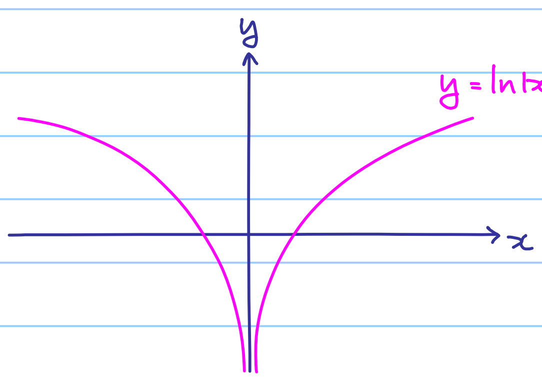
For $x < 0$, $y = \ln(-x)$

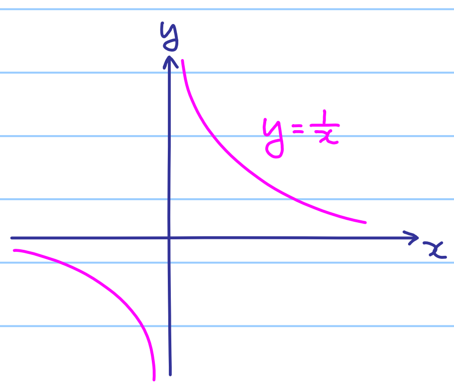
$$e^y = -x$$

$$e^y \frac{dy}{dx} = -1$$

$$\frac{dy}{dx} = \frac{-1}{e^y} = \frac{1}{x}$$

$\therefore \frac{d}{dx} \ln|x| = \frac{1}{x}$ for $x \neq 0$





Note: It is why $\int \frac{1}{x} dx = \ln|x| + C$.

putting absolute sign here.

e.g. Differentiation of Inverse Trigonometric Functions

Let $y = \sin^{-1}x$, $\sin^{-1}: [-1,1] \longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$. Then, $\sin y = x$.

differentiate both sides with respect to x

$$\cos y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$\sin y = x$, $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$\cos y = \pm\sqrt{1-\sin^2 y}$

$= \sqrt{1-x^2}$ or $-\sqrt{1-x^2}$

$\therefore \frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$

(rejected, $-\frac{\pi}{2} \le y \le \frac{\pi}{2} \Rightarrow \cos y \ge 0$)

Let $y = \cos^{-1}x$, $\cos^{-1}: [-1,1] \longrightarrow [0,\pi]$. Then, $\cos y = x$.

differentiate both sides with respect to x.

$$-\sin y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sin y}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

$\cos y = x$, $0 \le y \le \pi$

$\sin y = \pm\sqrt{1-\cos^2 y}$

$= \sqrt{1-x^2}$ or $-\sqrt{1-x^2}$

$\therefore \frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$

(rejected, $0 \le y \le \pi \Rightarrow \sin y \ge 0$)

Ex: Let $y = \tan^{-1}x$, $\tan^{-1}: \mathbb{R} \longrightarrow (-\frac{\pi}{2}, \frac{\pi}{2})$.

Find $\frac{dy}{dx}$. Ans: $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$

e.g. If $y = \sqrt[3]{\frac{(x-1)(x-2)^2}{x-4}}$, then find $\frac{dy}{dx}$.

Difficult to differentiate by using chain rule and quotient rule.

$$y^3 = \frac{(x-1)(x-2)^2}{x-4}$$

$$\ln y^3 = \ln \frac{(x-1)(x-2)^2}{x-4}$$

$$3\ln y = \ln(x-1) + 2\ln(x-2) - \ln(x-4)$$

$$\frac{3}{y}\frac{dy}{dx} = \frac{1}{x-1} + \frac{2}{x-2} - \frac{1}{x-4}$$

(Apply implicit differentiation)

$$\frac{dy}{dx} = \frac{y}{3}\left(\frac{1}{x-1} + \frac{2}{x-2} - \frac{1}{x-4}\right) = \frac{1}{3}\sqrt[3]{\frac{(x-1)(x-2)^2}{x-4}}\left(\frac{1}{x-1} + \frac{2}{x-2} - \frac{1}{x-4}\right)$$