THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1010G/H University Mathematics for Applications 2014-2015
Revision 2

Note: Questions will be discussed in lectures, no typed solution will be given.

1. Find the derivative of the following functions:
   (a) $f(x) = x^x$
   (b) $f(x) = x^{e^x}$

2. Evaluate the following integrals:
   (a) $\displaystyle\int \sin 2x \cos 6x \, dx$
   (b) $\displaystyle\int \sin^2 2x \sin 5x \, dx$
   (c) $\displaystyle\int \cos^5 x \sin^2 x \, dx$
   (d) $\displaystyle\int \sec^3 x \tan^3 x \, dx$
   (e) $\displaystyle\int \sec^6 x \tan x \, dx$
   (f) $\displaystyle\int \csc^6 x \cot^4 6x \, dx$
   (g) $\displaystyle\int \csc^5 x \cot^3 6x \, dx$

3. Evaluate the following integrals:
   (a) $\displaystyle\int \frac{1}{\sqrt{x^2 - 4x + 2}} \, dx$
   (b) $\displaystyle\int \frac{x}{\sqrt{4 - 4x - x^2}} \, dx$

4. Show by making an appropriate substitution that

$$\int_{\frac{1}{2}}^{2} \frac{1}{1 + x^3} \, dx = \int_{\frac{1}{2}}^{2} \frac{x}{1 + x^3} \, dx.$$

   Hence, evaluate $\displaystyle\int_{\frac{1}{2}}^{2} \frac{1}{1 + x^3} \, dx$.

5. Let n be a nonnegative integer and define

$$I_n = \int x^{n+\frac{1}{2}}\sqrt{1-x}\,dx.$$

Prove that

$$(2n+4)I_n = -2x^{n+\frac{1}{2}}(1-x)^{\frac{3}{2}} + (2n+1)I_{n-1}.$$

6. (a) Show that for any real numbers a and y, $e^y - e^a \geq e^a(y-a)$.

(b) By using the result in (a), prove that $\displaystyle\int_0^1 e^{x^2}\,dx \geq e^{\frac{1}{3}}$.

7. (a) Let n be a natural number. By using the identities

$$\frac{1}{1-t} = 1 + t + t^2 + \cdots + t^{n-1} + \frac{t^n}{1-t}$$

and

$$\frac{1}{1+t} = 1 - t + t^2 - \cdots + (-1)^{n-1}t^{n-1} + \frac{(-1)^n t^n}{1+t},$$

show that for any $x \in (-1,1)$,

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots - \frac{x^n}{n} - \int_0^x \frac{t^n}{1-t}\,dt$$

and

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} + \int_0^x \frac{(-1)^n t^n}{1+t}\,dt.$$

(b) Let k be a positive integer greater than 1. By using the result in (a), prove that for any $x \in (0,1)$,

$$0 \geq \ln(1+x)(1-x) + 2(\frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \cdots + \frac{x^{2k}}{2k}) \geq \frac{-2}{1-x^2}(\frac{x^{2k+2}}{2k+2}).$$

Hence, show that

$$\lim_{k\to\infty}\left[\frac{1}{2}(\frac{1}{2})^2 + \frac{1}{4}(\frac{1}{2})^4 + \frac{1}{6}(\frac{1}{2})^6 + \cdots + \frac{1}{2k}(\frac{1}{2})^{2k}\right]$$

exists and find its value.

8. (a) Let k be a natural number. Show that

$$\frac{1}{2k+1} < \int_k^{k+1} \frac{1}{2x-1}\,dx < \frac{1}{2k-1}.$$

Hence, show that

$$\frac{1}{2}\ln(2n+1) < 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1} < 1 + \frac{1}{2}\ln(2n-1),$$

where n is a natural number.

(b) By showing that

$$\frac{\sin^2 nx}{\sin x} = \sin x + \sin 3x + \cdots + \sin(2n-1)x,$$

prove that

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2 nx}{\sin x}\,dx = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}.$$

(c) Hence, show that

$$\lim_{n\to\infty}\frac{2}{\ln n}\int_0^{\frac{\pi}{2}} \frac{\sin^2 nx}{\sin x}\,dx = 1.$$