

Oblique asymptote

If $y = mx + c$ is a straight such that $\lim_{x \rightarrow +\infty} f(x) - (mx + c) = 0$, then the straight line is called an oblique asymptote of $f(x)$.

(Similar definition can be made for $-\infty$)



the distance tends to 0

as $x \rightarrow +\infty$

Suppose $y = mx + c$ is an oblique asymptote.

$$\text{i.e. } \lim_{x \rightarrow +\infty} f(x) - mx - c = 0$$

$$\text{Note: } \lim_{x \rightarrow +\infty} \frac{c}{x} = 0$$

$$\lim_{x \rightarrow +\infty} \frac{f(x)}{x} - m - \frac{c}{x} = \lim_{x \rightarrow +\infty} [f(x) - mx - c] \cdot \frac{1}{x} = 0 \cdot 0 = 0$$

$$\text{Then } \lim_{x \rightarrow +\infty} \frac{f(x)}{x} = \lim_{x \rightarrow +\infty} \left(\frac{f(x)}{x} - m - \frac{c}{x} \right) + \left(m + \frac{c}{x} \right) = 0 + m = m$$

i.e. if an oblique asymptote exists, the slope $m = \lim_{x \rightarrow +\infty} \frac{f(x)}{x}$ — (*)

Finding oblique asymptote

Compute $\lim_{x \rightarrow +\infty} \frac{f(x)}{x}$, define it to be m if exists.

Then, compute $\lim_{x \rightarrow +\infty} f(x) - mx$, define it to be c if exists.

If both limits exist, $y = mx + c$ is an oblique asymptote.

Remark:

1) If $m = 0$, it is just a horizontal asymptote, and in this case, $c = \lim_{x \rightarrow +\infty} f(x)$.

2) Even $\lim_{x \rightarrow +\infty} \frac{f(x)}{x}$ exists, we define it to be m .

$\lim_{x \rightarrow +\infty} f(x) - mx$ may NOT exist! Any example? (Think: $f(x) = \sqrt{x}$)

i.e. Converse of (*) is NOT true!

e.g. Let $f(x) = \frac{x|x-2|}{x-1}$, $x \neq 1$.

$$f(x) = \begin{cases} \frac{x(x-2)}{x-1} & \text{if } x \geq 2 \\ -\frac{x(x-2)}{x-1} & \text{if } x < 2 \text{ and } x \neq 1 \end{cases}$$

Ex: (a) Show that f is NOT differentiable at $x=2$.

Hint: Show that $\lim_{\Delta x \rightarrow 0} \frac{f(2+\Delta x) - f(2)}{\Delta x}$ does NOT exist.

$$(b) f'(x) = \begin{cases} \frac{x^2 - 2x + 2}{(x-1)^2} & \text{if } x > 2 \\ -\frac{x^2 - 2x + 2}{(x-1)^2} & \text{if } x < 2 \text{ and } x \neq 1 \end{cases}$$

Solve $f'(x) > 0$ and $f'(x) < 0$

Ans: $f'(x) > 0$ when $x > 2$

$f'(x) < 0$ when $x < 2$ and $x \neq 1$

min = (2, 0)

$$(c) f''(x) = \begin{cases} \frac{-2}{(x-1)^3} & \text{if } x > 2 \\ \frac{2}{(x-1)^3} & \text{if } x < 2 \text{ and } x \neq 1 \end{cases}$$

Solve $f''(x) > 0$ and $f''(x) < 0$

Ans: $f''(x) > 0$ when $1 < x < 2$

$f''(x) < 0$ when $x > 2$ or $x < 1$

point of inflection = (2, 0)

(d) vertical asymptote : $x=1$

oblique asymptote :

$$\lim_{x \rightarrow +\infty} \frac{f(x)}{x} = \lim_{x \rightarrow +\infty} \frac{x(x-2)}{x(x-1)} = 1 \quad \therefore m=1$$

$$\lim_{x \rightarrow +\infty} f(x) - mx = \lim_{x \rightarrow +\infty} \frac{x(x-2)}{x-1} - x = \lim_{x \rightarrow +\infty} \frac{-x}{x-1} = -1 \quad \therefore c=-1$$

oblique asymptote : $y=x-1$

Ex: How about $-\infty$? Ans: $y=-x+1$

(e) x -intercept: Solve $f(x)=0$

$$\frac{x|x-2|}{x-1} = 0$$

$$x = 0 \text{ or } 2$$

y -intercept: $f(0) = 0$.

(f) Sketch $y=f(x)$.

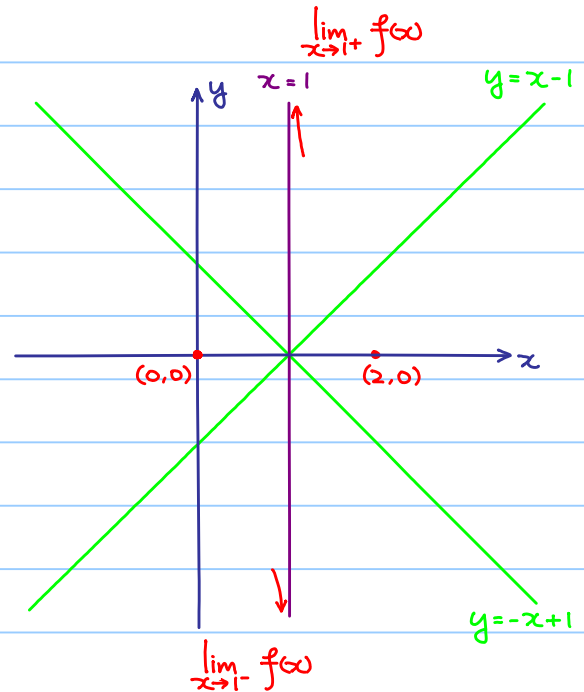
Step 1: draw asymptotes

Step 2: put down x -intercepts
and y -intercept

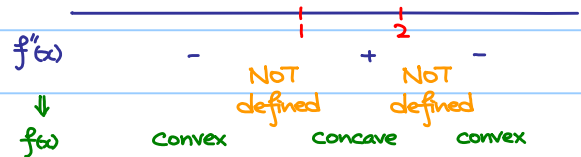
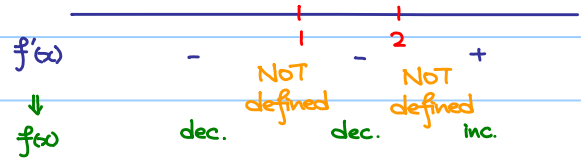
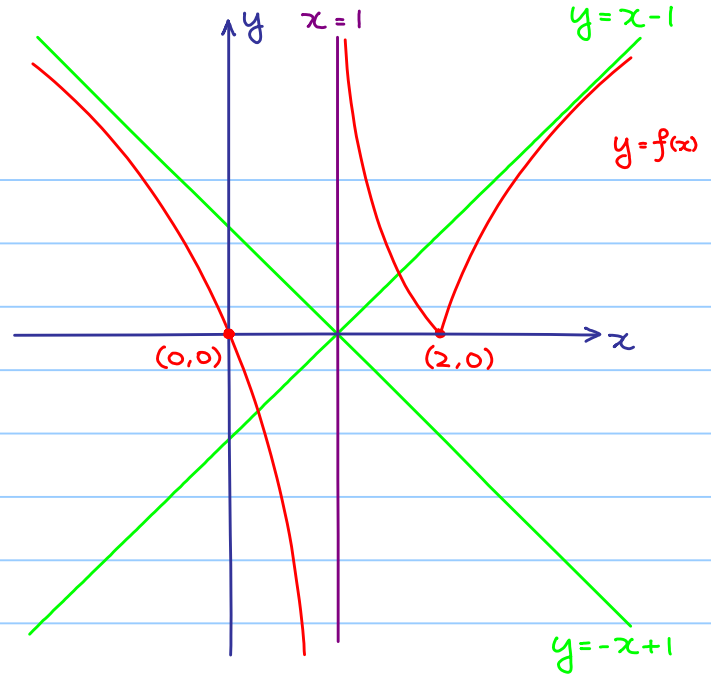
Step 3:

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} -\frac{x(x-2)}{x-1} = -\infty$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} -\frac{x(x-2)}{x-1} = +\infty$$



Step 4: Use the information $f'(x)$ and $f''(x)$



Curve Sketching :

Goal: Given a function $f(x)$, sketch the graph of $y=f(x)$.

(Capturing main features)

- x-intercept
- y-intercept
- increasing / decreasing
saddle point / max. / min.
- concave / convex
point of inflection
- vertical asymptote
- horizontal asymptote
- oblique asymptote

$$\text{solve } f(x) = 0$$

$$\text{y-intercept} = f(0)$$

$$\text{solve } f'(x) > 0 / f'(x) < 0$$

change of sign of $f'(x)$?

$$\text{solve } f''(x) > 0 / f''(x) < 0$$

change of sign of $f''(x)$?

any $x=a$ with $\lim_{x \rightarrow a^+} f(x) = \pm\infty$ or $\lim_{x \rightarrow a^-} f(x) = \pm\infty$

$$m = \lim_{x \rightarrow +\infty} \frac{f(x)}{x}$$

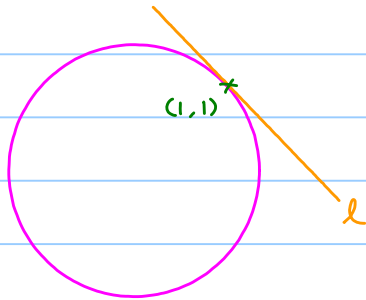
$$c = \lim_{x \rightarrow +\infty} f(x) - mx$$

Implicit Differentiation

e.g. $x^2 + y^2 = 2$ — \mathcal{C}

Locus of \mathcal{C} is a circle centered at $(0,0)$ with radius $\sqrt{2}$.

Check: $(1,1)$ is a point lying on the circle.

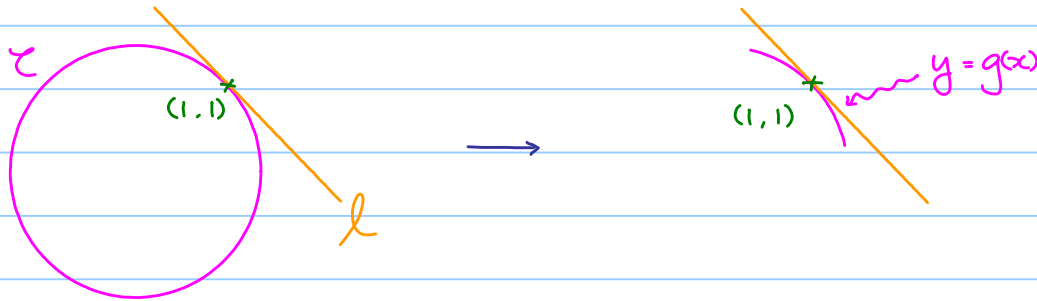


We want to find the equation of the tangent line l
(i.e. need to know the slope of l)

Note: $x^2 + y^2 = 2$ is NOT a function.

Question: How to find $\frac{dy}{dx}$? (and, actually, is it defined?)

Answer: Yes, roughly speaking,



The small segment of \mathcal{C} containing $(1, 1)$ can be regarded as the graph of some function $y = g(x)$. (In fact, $y = \sqrt{2-x^2}$ in this case.)

How to find? Do it as usual?

eg. $x^2 + y^2 = 2$

differentiate both sides with respect to x .

$$2x + \frac{d}{dx} y^2 = 0$$

$$2x + 2y \frac{dy}{dx} = 0 \quad (\text{Applying chain rule})$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

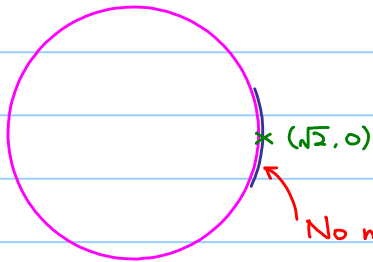
$$\therefore \frac{dy}{dx} = -1 \quad \text{when } (x, y) = (1, 1).$$

$$\text{We denote it by } \left. \frac{dy}{dx} \right|_{(x, y) = (1, 1)} = -1$$

Remark :

$\frac{dy}{dx}$ is defined at a point of a curve only if a small arc containing the point can be regarded as the graph of some function $y=g(x)$.

$\therefore \frac{dy}{dx}$ is NOT defined when $(x,y) = (\pm\sqrt{2}, 0)$.



No matter how small the arc is,

it cannot be realized as graph of some function $y=g(x)$.

$$\text{e.g. } x^3 + y^3 - 3xy = 0 \quad \text{--- } \mathcal{C}$$

$$3x^2 + 3y^2 \frac{dy}{dx} - 3y - 3x \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{y-x^2}{y^2-x}$$

If we want to find the slope
of the tangent line at A .

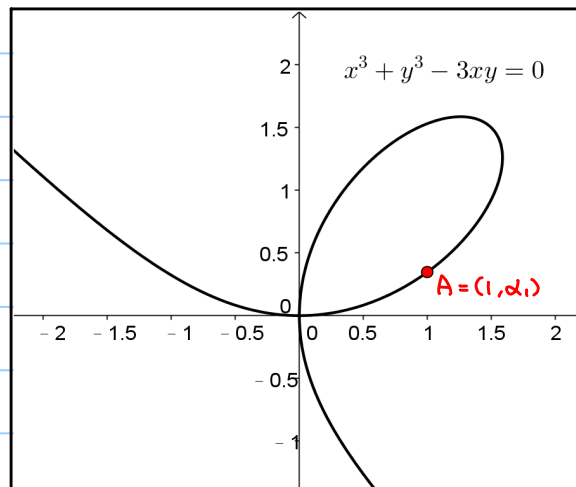
putting $x=1$ into \mathcal{C} .

$$y^3 - 3y + 1 = 0$$

↳ NOT easy to solve!

FACT: The above equation has three roots, two roots α_1, α_2 are positive ($\alpha_1 < \alpha_2$)
one root is negative.

$A = (1, \alpha_1)$ and what we need is $\left. \frac{dy}{dx} \right|_{(x,y)=(1,\alpha_1)}$



Applications :

e.g. Differentiation of Logarithmic Function

Let $y = \ln x$, $x > 0$. Then $e^y = x$,

differentiate both sides with respect to x .

$$e^y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

$$\therefore \frac{d}{dx} \ln x = \frac{1}{x} \text{ for } x > 0.$$

Ex: By rewriting $\log_a x = \frac{\ln x}{\ln a}$, show that $\frac{d}{dx} \log_a x = \frac{1}{x \ln a}$.

e.g. Let $y = \ln|x|$, $x \neq 0$. Find $\frac{dy}{dx}$.

We can rewrite $y = \begin{cases} \ln x & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}$

For $x > 0$, we have just shown that $\frac{dy}{dx} = \frac{1}{x}$

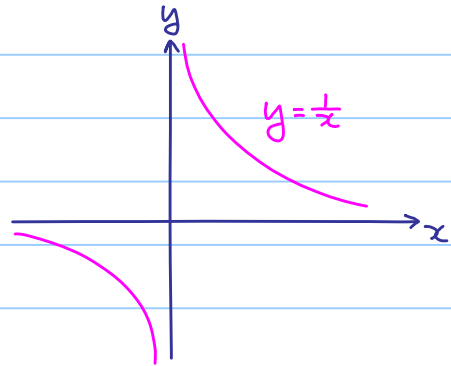
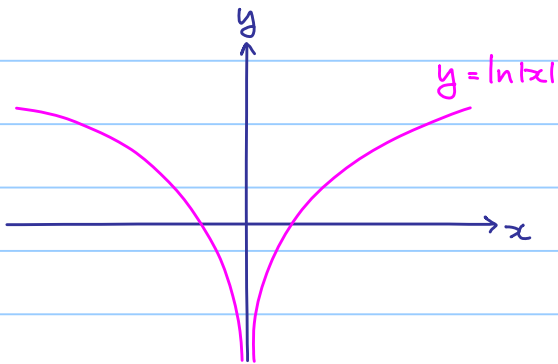
For $x < 0$, $y = \ln(-x)$

$$e^y = -x$$

$$e^y \frac{dy}{dx} = -1$$

$$\frac{dy}{dx} = \frac{-1}{e^y} = \frac{1}{x}$$

$\therefore \frac{d}{dx} \ln|x| = \frac{1}{x}$ for $x \neq 0$



Note: It is why $\int \frac{1}{x} dx = \ln|x| + C$.
↑ putting absolute sign here.

e.g. Differentiation of Inverse Trigonometric Functions

Let $y = \sin^{-1}x$, $\sin^{-1}: [-1, 1] \rightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$. Then, $\sin y = x$.

differentiate both sides with respect to x .

$$\cos y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\sin y = x, \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

$$\cos y = \pm \sqrt{1-\sin^2 y}$$

$$= \sqrt{1-x^2} \quad \text{or} \quad -\sqrt{1-x^2}$$

$$\therefore \frac{d}{dx} \sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$$

(rejected, $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \Rightarrow \cos y \geq 0$)

Let $y = \cos^{-1}x$, $\cos^{-1}: [-1, 1] \rightarrow [0, \pi]$. Then, $\cos y = x$.

differentiate both sides with respect to x .

$$-\sin y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sin y}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

$$\cos y = x, \quad 0 \leq y \leq \pi$$

$$\sin y = \pm \sqrt{1-\cos^2 y}$$

$$= \sqrt{1-x^2} \quad \text{or} \quad -\sqrt{1-x^2}$$

$$\therefore \frac{d}{dx} \cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$$

(rejected, $0 \leq y \leq \pi \Rightarrow \sin y \geq 0$)

Ex: Let $y = \tan^{-1}x$, $\tan^{-1}: \mathbb{R} \rightarrow (-\frac{\pi}{2}, \frac{\pi}{2})$.

Find $\frac{dy}{dx}$. Ans: $\frac{d}{dx} \tan^{-1}x = \frac{1}{1+x^2}$

e.g. If $y = \sqrt[3]{\frac{(x-1)(x-2)^2}{x-4}}$, then find $\frac{dy}{dx}$.

Difficult to differentiate by using chain rule and quotient rule.

$$y^3 = \frac{(x-1)(x-2)^2}{x-4}$$

$$\ln y^3 = \ln \frac{(x-1)(x-2)^2}{x-4}$$

$$3 \ln y = \ln(x-1) + 2 \ln(x-2) - \ln(x-4)$$

$$\frac{3}{y} \frac{dy}{dx} = \frac{1}{x-1} + \frac{2}{x-2} - \frac{1}{x-4}$$

(Apply implicit differentiation)

$$\frac{dy}{dx} = \frac{y}{3} \left(\frac{1}{x-1} + \frac{2}{x-2} - \frac{1}{x-4} \right) = \frac{1}{3} \sqrt[3]{\frac{(x-1)(x-2)^2}{x-4}} \left(\frac{1}{x-1} + \frac{2}{x-2} - \frac{1}{x-4} \right)$$