Notes 4.

Implicit Differentiation. & 2^{nd} Derivative Test.

Intro Previously, we mentioned the $+, -, \times, \div$ of derivatives. next we mention (but not 'prove'!) the important

'Implicit differentiation'!

We have 2 steps

1) (a) given a 'nice' fn. of 2 variables $F(x,y)$ & consider the equation

$$F(x,y) = C$$

L.H.S. = fn. of 2 variables

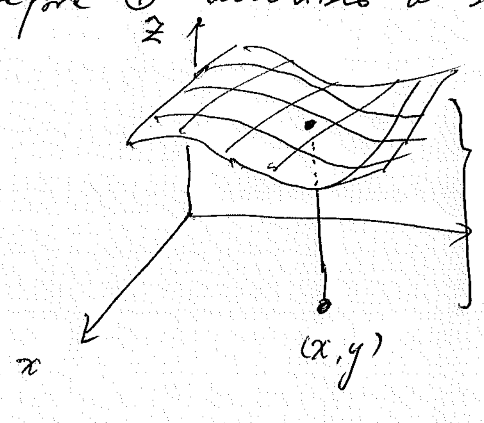
R.H.S. = const.

then we can interpret / understand (理解為) it as:

the simultaneous (公共解) solution of the 2 equations

$$z = F(x,y) \quad \text{———} \quad ①$$
$$z = c \quad \text{———} \quad ②$$

(b) Now, just a fn. of 1 variable $y = f(x)$ leads to a curve, a fn. of 2 variable leads to a surface, therefore ① describes a surface.

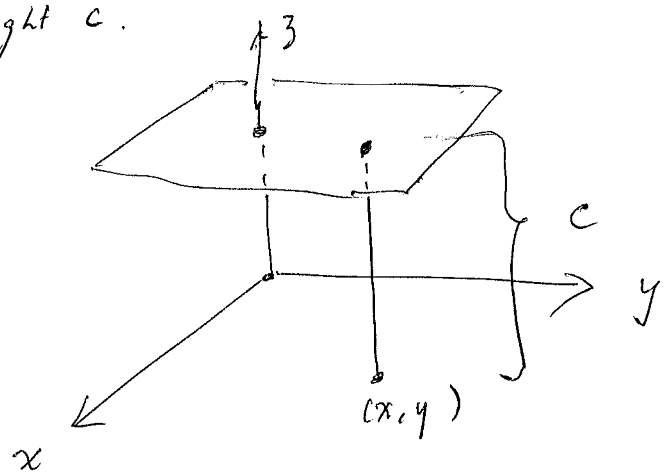


$f(x,y) = z$ = height above the point (x,y).

<u>Next</u>, the equation (2) simply means

"height $= z = c$ for any choice of (x, y)".

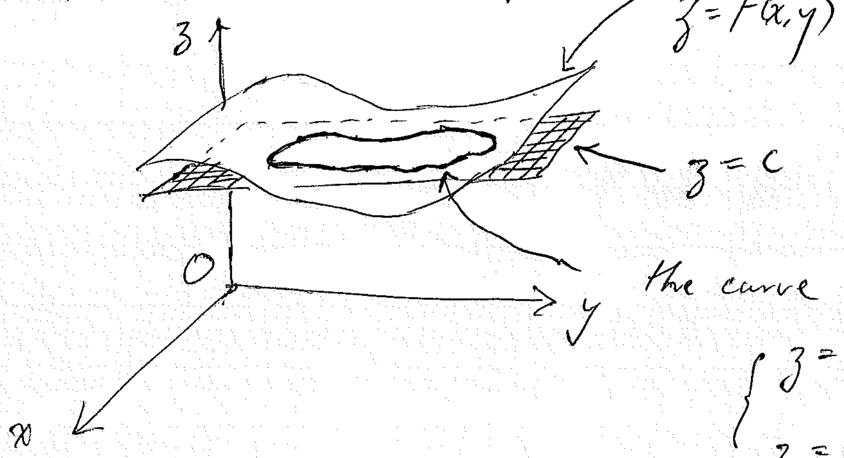
I.e. it describes a plane parallel to the xy-plane at height c.



Combining the above $\begin{cases} z = F(x,y) \\ z = c \end{cases}$

means "the <u>intersection</u> of the <u>surface</u> $z = F(x,y)$ with the plane $z = c$".

hence it describes curves.



the curve given by

$$\begin{cases} z = F(x,y) \\ z = c \end{cases}$$

or $F(x,y) = c$.

2) From 1), we see that $F(x,y) = C$ describes curve(s) (may be more than 1 curve!), therefore

y is a fn. of x
or x is a fn. of y!

I.e. $F(x,y) = C$

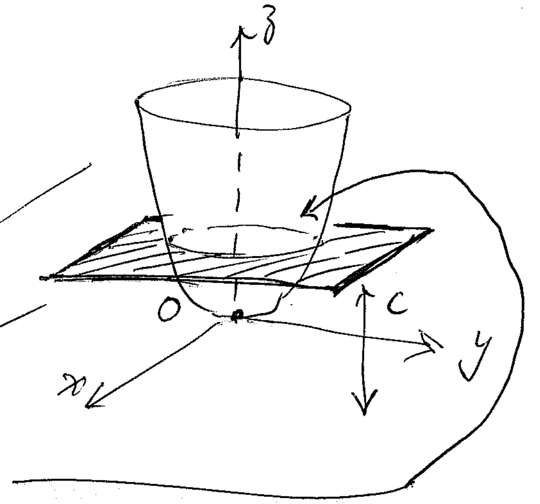
means 'actually' $F(x, y(x)) = C$ ← $y = \tilde{y}(x)$ — fn. of x (we don't write $f(x)$ to be economical!)

or $F(x(y), x) = C$ ← $x = x(y)$ — fn. of y

Examples: ① $F(x,y) = x^2+y^2 = C$

We can interpret this as:

$$\begin{cases} z = x^2+y^2 \\ z = C \end{cases}$$

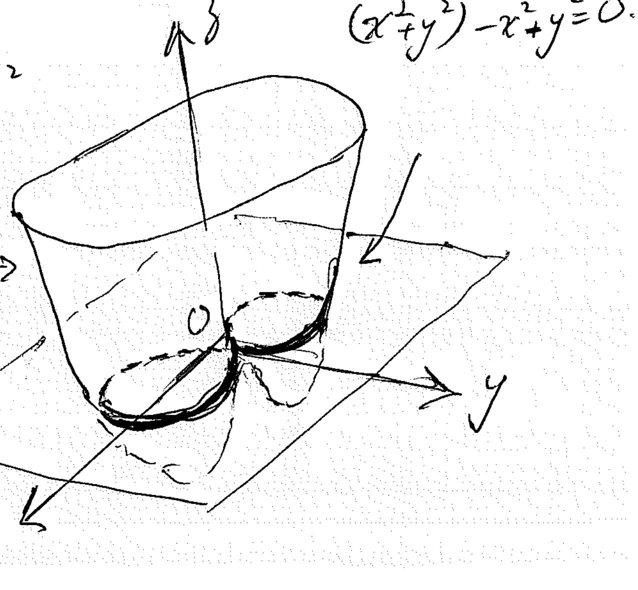


Their intersection is a circle of radius $\sqrt{C}$.

② $F(x,y) = (x^2+y^2)^2 - x^2+y^2$

$z = (x^2+y^2)^2 - x^2+y^2$ ⟶

$z = 0$ ⟶



Summary. • An equation of the form (Rk: Actually, we need some assumptions on F, but we'll omit them here!)

$$F(x,y) = c. \quad \text{——— } (*)$$

describes curve(s). (In geography, such curves are known as contour curves.)

• Because (*) describes 'curves', therefore $y = y(x)$ or $x = x(y)$.

Example: (1) $\underbrace{x^2+y^2}_{F(x,y)} = c$

$$\Rightarrow \quad y = \sqrt{c-x^2} \quad \text{or} \quad y = -\sqrt{c-x^2}$$

$$\text{or} \quad x = \sqrt{c-y^2} \quad \text{or} \quad x = -\sqrt{c-y^2}$$

(2) $\underbrace{(x^2+y^2)^2 - x^2+y^2}_{F(x,y)} = 0$

$$\Rightarrow \quad x^4 + 2x^2y^2 + y^4 - x^2 + y^2 = 0$$

$$\Rightarrow \quad y^4 + y^2(2x^2+1) + x^4 - x^2 = 0$$

$$\Rightarrow \quad y^2 = \frac{-(2x^2+1) \pm \sqrt{(2x^2+1)^2 - 4(x^4-x^2)}}{2}$$

$$\Rightarrow \quad y = \pm\sqrt{\cdots\cdots}$$

Similarly, x represents functions of y.

$\square$

Implicit Differentiation

From the above, we know that the equation

$$F(x,y) = c$$

implies $y = y(x)$ or $x = x(y)$

therefore, we can compute $y' = y'(x)$ or $x' = x'(y)$.

(Rk: " ' " means differentiating w.r.t. the variable in the brackets! I.e. $y'(t) = \frac{dy(t)}{dt}$, $y'(x) = \frac{dy(x)}{dx}$, $y'(x^2) = \frac{dy(x^2)}{dx^2}$)

Examples.

1). Compute $y'(x)$ for

$$x^2 + xy + y \cdot e^x = 10$$

Solution.

Differentiating both sides of the above equation w.r.t. x gives

$$\frac{d}{dx}(x^2 + xy + ye^x) = \frac{d10}{dx} = 0$$

implicit fn. of x!

Implying $\frac{dx^2}{dx} + \frac{d}{dx}(xy) + \frac{d(ye^x)}{dx} = 0$

product rule product rule!

$$\Rightarrow \quad 2x + \boxed{x\frac{dy}{dx} + \frac{dx}{dx}y} + \boxed{y\frac{de^x}{dx} + \frac{dy}{dx}e^x} = 0$$

$$\Rightarrow \quad 2x + xy' + y + ye^x + y'e^x = 0$$

$$\Rightarrow \quad y'(x+e^x) = -2x - y - ye^x$$

$$\boxed{\text{Ans: } y' = -\frac{(2x+y+ye^x)}{x+e^x}}$$

#

2) Compute y'' for $x^2 + xy = 10$.

Sol. $$\frac{d}{dx}(x^2 + xy) = \frac{d\,10}{dx} = 0$$

$$\Rightarrow 2x + xy' + y = 0 \quad \text{—— (1)} \quad \left(\text{i.e. } y' = \frac{-y-2x}{x}\right)$$

Differentiate again

$$\Rightarrow \frac{d}{dx}(2x + xy' + y) = \frac{d0}{dx} = 0$$

product rule

$$\Rightarrow 2 + \boxed{y' + xy''} + y' = 0$$

$$\Rightarrow xy'' + 2y' + 2 = 0$$

$$\Rightarrow xy'' + 2\left(\frac{-y-2x}{x}\right) + 2 = 0$$

$$\Rightarrow y'' = \left[-2 + 2\left(\frac{y+2x}{x}\right)\right] \Big/ x \qquad \#$$

Mean Value Theorems.

We try to be brief and summarize only the 3 mean value theorems here, with proofs omitted. (we'll provide them next!)

Thm. 1) (Rolle's Thm.)

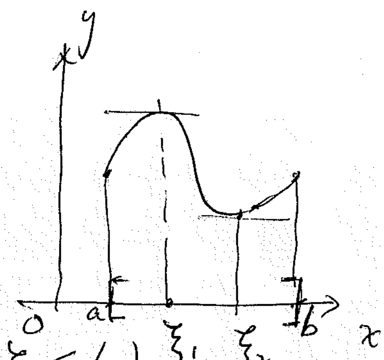
Assumptions: (1) $f : [a,b] \to \mathbb{R}$ cont. (closed)

(2) $f : (a,b) \to \mathbb{R}$ open (open)

(3) $f(a) = f(b)$

Claim: $\exists\, \xi$ between a & b (i.e. $a < \xi < b$)

s.t. $f'(\xi) = 0$.



(Rk: There may be more than 1 such points!) 2 such points.

Thm 2) Lagrange's Mean Value Thm. (LMVT)

Assumptions: ① $f : [a, b] \longrightarrow \mathbb{R}$ cont.,

② $f : (a, b) \longrightarrow \mathbb{R}$ diff.

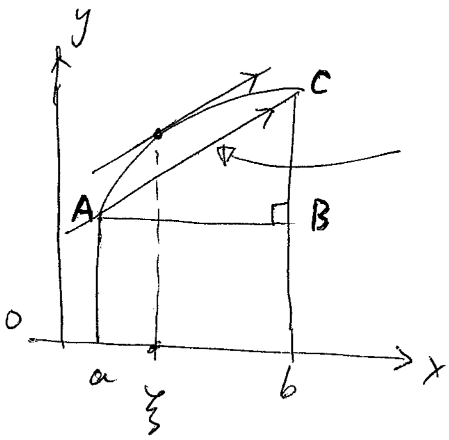
Claim :- $\exists \xi \in (a, b)$ s.t.

$$\frac{f(b) - f(a)}{b - a} = f'(\xi).$$

Rk: If we call the L.H.S. the 'average value' or the 'mean value' of f, then the theorem says.

"the mean value is attained by f' at some point ξ!"

Picture:



$\frac{f(b) - f(a)}{b - a}$ = slope of this line!

Rk: Geometrically, the LMVT says

" the slope $\frac{f(b) - f(a)}{b - a}$ of the line AC

is attained by f' at some point ξ ! "

Thm 3 (Cauchy Mean Value Thm) (CMVT)

Assump: ① $f : [a,b] \longrightarrow \mathbb{R}$, $g : [a,b] \longrightarrow \mathbb{R}$ both cont.,

② $f : (a,b) \longrightarrow \mathbb{R}$, $g : (a,b) \longrightarrow \mathbb{R}$ both diff.,

③ (Technical assumption to avoid $g(b) = g(a)$!)

$$g'(x) \neq 0 \quad \forall x \in (a,b)$$

then $\exists \xi \in (a,b)$ s.t. $\dfrac{f(b) - f(a)}{g(b) - g(a)} = \dfrac{f'(\xi)}{g'(\xi)}$.

Rk: If $g(x) = x \quad \forall x \in (a,b)$, we get back the L.M.V.T. because in this case

$$g'(x) = \frac{dx}{dx} = 1 \qquad \forall x \in (a,b)$$

and therefore

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f(b) - f(a)}{b - a} = \frac{f'(\xi)}{g'(\xi)} = f'(\xi)$$

$$g'(\xi) = \left.\frac{dg(x)}{dx}\right|_{x=\xi} = \left.\frac{dx}{dx}\right|_{x=\xi} = 1$$

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L'Hôpital Rule

One nice application of the mean value theorems is the L'Hôpital Rule. (In the following, we use "L'H" to mean L'Hôpital Rule!)

There are many cases of the L'Hôpital Rule, which helps to compute limits of the forms:

$$\lim_{x\to c} \frac{f(x)}{g(x)} \qquad \left(= \frac{0}{0}\right)$$

$$\left(= \frac{\infty}{\infty}\right) \qquad \text{or } \frac{-\infty}{-\infty} \quad \text{etc.}$$

$$\text{or } \lim_{x\to +\infty} \frac{f(x)}{g(x)} \qquad \left(= \frac{0}{0}\right)$$

$$\left(= \frac{\infty}{\infty}\right) \qquad \text{or } \frac{-\infty}{-\infty} \quad \text{etc.}$$

$$\text{or } \lim_{x\to -\infty} \frac{f(x)}{g(x)} \qquad \left(= \frac{0}{0}\right)$$

$$\left(= \frac{\infty}{\infty}\right) \qquad \text{or } \frac{-\infty}{-\infty} \quad \text{etc.}$$

We just prove a simple case:

Thm. (L'H Rule)

Assump.: ① $f : (a,b) \longrightarrow \mathbb{R}$, $g : (a,b) \longrightarrow \mathbb{R}$ both diff.

② $f(c) = 0$, $g(c) = 0$ $\quad \exists\, c \in (a,b)$.

③ $g'(x) \neq 0 \quad \forall x \in (a,b)$

④ $\displaystyle\lim_{x\to c} \frac{f'(x)}{g'(x)} = L$ for some finite number L.

Claim:- $\displaystyle\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)} = L$

Proof: Rewrite $\frac{f(x)}{g(x)}$ in the form $\frac{f(x) - \overset{=0}{f(c)}}{g(x) - \underset{=0}{g(c)}}$

hence $\frac{f(x)}{g(x)} = \frac{f(x) - f(c)}{g(x) - g(c)}$.

Apply L'H to $\frac{f(x) - f(c)}{g(x) - g(c)}$ to get

$$\frac{f(x) - f(c)}{g(x) - g(c)} = \frac{f'(\xi)}{g'(\xi)}$$ $\exists\, \xi$ between x & c

i.e. $\xi \in (x, c)$

or $\xi \in (c, x)$

depending on $x < c$

or $c < x$.

Next, let $x \to c$, the $\xi \to c$ too (because ξ is between x & c)

therefore $\lim_{x\to c} \frac{f(x) - f(c)}{g(x) - g(c)} = \lim_{x\to c} \frac{f'(\xi)}{g'(\xi)} = \lim_{\xi \to c} \frac{f'(\xi)}{g'(\xi)} = L$.

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Examples 1). Find $\lim_{x\to 0} \frac{\sin x}{x}$

Sol.: $\lim_{x\to 0} \frac{\sin x}{x}$ is of the form $\frac{0}{0}$!

hence by L'H $\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\frac{d \sin x}{dx}}{\frac{dx}{dx}} = \lim_{x\to 0} \frac{\cos x}{1} = 1$ #

2) Find $\lim_{x\to\infty} \frac{x^3}{e^x}$

Sol.: $\lim_{x\to\infty} \frac{x^3}{e^x}$ is of the form $\frac{\infty}{\infty}$. therefore

We apply L'H Rule to get

$$\lim_{x\to\infty} \frac{x^3}{e^x} = \lim_{x\to\infty} \frac{\frac{dx^3}{dx}}{\frac{de^x}{dx}} = \lim_{x\to\infty} \frac{3x^2}{e^x}$$ (again of the form $\frac{\infty}{\infty}$!)

We apply L'H again to get

$$\lim_{x\to\infty} \frac{3x^2}{e^x} = \lim_{x\to\infty} \frac{\frac{d3x^2}{dx}}{\frac{de^x}{dx}} = \lim_{x\to\infty} \frac{6x}{e^x}$$ (again of the form $\frac{\infty}{\infty}$!)

We apply L'H again to get

$$\lim_{x\to\infty} \frac{6x}{e^x} = \lim_{x\to\infty} \frac{\frac{d6x}{dx}}{\frac{de^x}{dx}} = \lim_{x\to\infty} \frac{6}{e^x} = 0$$

Concl: $\lim_{x\to\infty} \frac{x^3}{e^x} = 0$ ⌗

Second Derivative Test.

There are two more applications of the mean value theorems, They are
① Taylor Theorem (see next notes)

② Second Derivative Test.

We mention here only the Second Derivative Test. (Proof Omitted!)

Thm:

Assump. ① $f: (a,b) \to \mathbb{R}$ diff. (i.e. $f'(x)$ exists $\forall x \in (a,b)$)

② $f''(c)$ exists at $c \in (a,b)$

③ $f'(c) = 0$ & $f''(c) > 0$

Claim:- c is a local minimum point.

(i.e. $f(c) \le f(x)$ $\forall x$ sufficiently near c)