Notes 3

Further Differentiation Rules.

In the following, we will prove

(i) Quotient Rule

& (ii) Chain Rule.

Quotient Rule 1 | Let $g: (a,b) \longrightarrow \mathbb{R}$, $c \in (a,b)$

be (i) differentiable at $c \in (a,b)$;

(ii) $g(c) \neq 0$.

then $\left(\frac{1}{g}\right)'(c) = \frac{-g'(c)}{g^2(c)}$ ← $\boxed{\text{Simplest Case}}$

Proof: Consider $\frac{\Delta(\frac{1}{g})}{\Delta x} = \frac{\frac{1}{g}(x) - \frac{1}{g}(c)}{x-c}$, where $x \neq c$

$$= \frac{\frac{1}{g(x)} - \frac{1}{g(c)}}{x-c} = \frac{g(c) - g(x)}{(x-c)g(x)g(c)}$$

$$= -\left(\frac{g(x)-g(c)}{x-c}\right)\frac{1}{g(x)g(c)} \qquad \text{———} \quad (1)$$

Next, let $x \to c$ and consider the limit $\lim_{x\to c} \frac{\frac{1}{g}(x) - \frac{1}{g}(c)}{x-c}$.

Because of (1), we know that (I) $\lim_{x\to c} \frac{g(x)-g(c)}{x-c} = g'(c)$

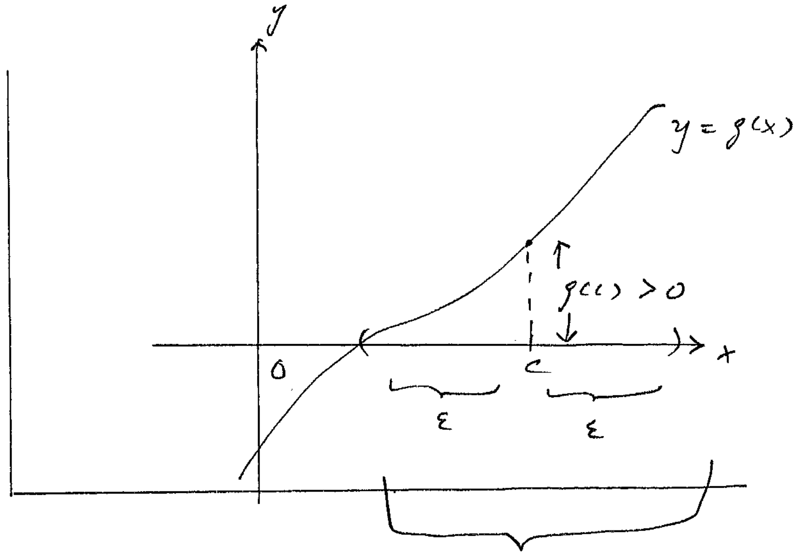
(by differentiability of g at $x = c$).

(II) $\lim_{x\to c} \frac{1}{g(c)} = \frac{1}{g(c)}$ ($\because$ $\frac{1}{g(c)}$ is a const. fn.)

(III) Since $g(c) \neq 0$, "$\underbrace{\exists \varepsilon > 0 \text{ such that } \forall x \in (c-\varepsilon, c+\varepsilon)}$

$\boxed{\text{There exists an } \varepsilon > 0 \text{ s.t. for all } x \text{ whose distance from } c \text{ is less than } \varepsilon.}$

it holds that $g(x) \neq 0$." ← We didn't prove this statement! But it's true! see picture below!



In this open interval, $g(x)$ is greater than 0.

Rk: Similar statement holds, if ">0" is replaced by "<0".

(Proof continued):

(IV) Since in the interval $(c-\varepsilon, c+\varepsilon)$, $g(x) \neq 0$, $\therefore \frac{1}{g(x)}$ is defined.

Now, as $x \to c$, $\lim_{x \to c} \frac{1}{g(x)} = \frac{1}{g(c)}$ (by continuity of g at $x = c$)

$\because$ g is diff. at $x = c$

$\therefore$ g is cont. at $x = c$.

(V) Summarizing all the above, we have

$$\lim_{x \to c} \frac{\frac{1}{g}(x) - \frac{1}{g}(c)}{x - c} = -\lim_{x \to c}\left(\frac{g(x) - g(c)}{x - c}\right) \frac{1}{\lim_{x \to c} g(x) g(c)}$$

$$= -g'(c) \cdot \frac{1}{g^2(c)}$$

$$= -\frac{g'(c)}{g^2(c)}$$

Hence $\left.\frac{d\left(\frac{1}{g}\right)(x)}{dx}\right|_{x=c} = \frac{\left.\frac{dg(x)}{dx}\right|_{x=c}}{g^2(c)}$ $\square$

Quotient Rule 2 Let $f : (a,b) \longrightarrow \mathbb{R}$ be diff. at $c \in (a,b)$;
$g : (a,b) \longrightarrow \mathbb{R}$ be diff. at $c \in (a,b)$;
$g(c) \neq 0$. Then

$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)\,g(c) - g'(c)\,f(c)}{g^2(c)}$$

Proof: Let $k(x) = \frac{1}{g}(x)$.

Then by the Product Rule

$$(f \cdot k)'(c) = f'(c)\,k(c) + k'(c) \cdot f(c) \qquad \text{——— } ①$$

But $k(c) = \dfrac{1}{g(c)}$; $k'(c) = \left(\dfrac{1}{g}\right)'(c) = \dfrac{-g'(c)}{g^2(c)}$

therefore ① becomes

$$\left(f \cdot \frac{1}{g}\right)'(c) = f'(c) \cdot \frac{1}{g(c)} + \frac{-g'(c)}{g^2(c)}\,f(c)$$

$$= \frac{g(c)\,f'(c) - g'(c)\,f(c)}{g^2(c)} \qquad \sim \square$$

Chain Rule.

- The idea of composite fn. (複合函數)

Example: Consider the fn. $\sqrt{1+x}$, it is actually the 2 rules $x \xmapsto{f} 1+x$

& $y \xmapsto{g} \sqrt{y}$

combined together by ① applying f to x first, obtaining $f(x) = 1+x$

(ii) give the name y to the result, i.e. $y = f(x) = \cancel{}\ 1+x$

(iii) Apply the rule g to y.

The picture is:

$$\begin{array}{ccccc} x & \xmapsto{f} & 1+x & \longmapsto & \sqrt{1+x} \\ & & \| & & \\ & & y & \xmapsto{g} & \sqrt{y} \end{array}$$

If we write $g \circ f$ (read "g circle f") for the combined rule, we obtain

$$x \xmapsto{g \circ f} \sqrt{1+x}$$

i.e. $$(g \circ f)(x) = g(f(x))$$

I put a bracket here, but it's not necessary!

Now, we can describe the Chain Rule!

Chain Rule Let g and f be real fns.

Suppose (i) f is diff. at $x = c$

Let $y = f(x)$, $y_0 = f(c)$

(ii) g is diff. at y_0

Then $g \circ f$ is diff. at $x = c$ <u>and</u>

$$(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$$

or equivalently $$\left.\frac{d(g \circ f)(x)}{dx}\right|_{x=c} = \left.\frac{d\,g(y)}{dy}\right|_{y=f(c)} \cdot \left.\frac{d f(x)}{dx}\right|_{x=c}$$

Proof: Consider $\dfrac{\Delta}{\Delta x} g\circ f = \dfrac{(g\circ f)(x) - (g\circ f)(c)}{x-c}$, $x \neq c$

$$= \frac{g(f(x)) - g(f(c))}{x-c}$$

which can be re-written in the form

$$= \frac{g(f(x)) - g(f(c))}{f(x) - f(c)} \; \frac{f(x) - f(c)}{x-c}$$

Now if as $x \to c$, $f(x) \neq f(c)$, then we can take limit to obtain

$$\lim_{x\to c} \frac{(g\circ f)(x) - (g\circ f)(c)}{x-c} = \underbrace{\lim_{x\to c} \frac{g(f(x)) - g(f(c))}{f(x) - f(c)}}_{I} \; \underbrace{\lim_{x\to c} \frac{f(x) - f(c)}{x-c}}_{II}$$

Note that ⓐ The limit (II) exists + equals $f'(c)$.

ⓑ As for the limit I, since $y = f(x)$, $y_0 = f(c)$,

$$\frac{g(f(x)) - g(f(c))}{f(x) - f(c)} = \frac{g(y) - g(y_0)}{y - y_0} \quad \text{——— } ①$$

Now, since f is cont. at $x=c$, ($\because$ f is diff. at $x=c$!)

we have $\lim_{x\to c} f(x) = f(c)$ or equivalently

$\lim_{x\to c} y = y_0$ or equivalently

$x \to c \Rightarrow y \to y_0$

Hence $$\lim_{x\to c} \frac{g(f(x)) - g(f(c))}{f(x) - f(c)} = \lim_{y\to y_0} \frac{g(y) - g(y_0)}{y - y_0} \quad \text{——— } ②$$

$$= g'(\underbrace{f(c)}_{y_0})$$

Hence the limit I is $g'(f(c))$.

Combining all the above, we have

$$\lim_{x\to c} \frac{g(f(x)) - g(f(c))}{x-c} = g'(f(c)) \cdot f'(c)$$

where $'$ means "differentiating with respect to the independent variable in question".

e.g. $g'(f(c))$ = differentiating g with respect to the independent variable y & calculate the answer at $y = f(c) = y_0$

$(g\circ f)'(c)$ = differentiating $g\circ f$ with respect to the independent variable x & calculate the answer at $x = c$.

Next case, Q: How about the case $x \to c$ but $f(x) = f(c)$?
A: In this case

$$\frac{g\circ f(x) - g\circ f(c)}{x-c} = 0 \text{ already, implying}$$

$$\lim_{x\to c} \frac{g\circ f(x) - g\circ f(c)}{x-c} = (g\circ f)'(c) = 0$$

As for the R.H.S. (i.e. right-hand side) of the Chain Rule,

(I) the term
"$g'(f(c))$ = ~~the~~ derivative of g calculated at $y = f(c) = y_0$"
is a finite no. ($\because$ we're assuming "g is diff. at y_0")

(II) the term $f'(c) = \lim_{x\to c} \frac{f(x) - f(c)}{x-c} = 0$ ($\because$ $x \to c$ but $f(x) = f(c)$)

Hence R.H.S. = $g'(f(c)) \cdot f'(c)$
= finite no. $\times 0 = 0$ = L.H.S. = $g'(f(c))$

Therefore we have proven the Chain Rule. $\square$