No-arbitrage Pricing Approach and Fundamental Theorem of Asset Pricing

presented by

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Parable of the bookmaker

- Taking bets on a two-horse race.

- The bookmaker calculates that one horse has a 25% chance of winning and the other a 75% chance. The odds are set at $3-1$ against.

Note on odds $n-m$ against eg $3-1$ against (一賠三)

A successful bet of \$$m$ will be awarded with \$$n$ plus stake returned.

- The implied probability of victory is $m/(m+n)$.

- It is equivalent to $m-n$ on.

Actual probability	25%	75%	
Bets placed	\$5000	\$10000	
1. Quoted odds	$13 - 5$ against	$15 - 4$ on	
Implied probability	28%	79%	Total = 107%
Profit if horse wins	−\$3000	\$2333	Expected profit = \$1000
2. Quoted odds	$9 - 5$ against	$5 - 2$ on	
Implied probability	36%	71%	Total = 107%
Profit if horse wins	\$1000	\$1000	Expected profit = \$1000

Forward contract 期貨合約

The buyer of the forward contract agrees to pay the delivery price K dollars at future time T to purchase a commodity whose value at time T is S_T. The pricing question is how to set K?

How about

$$E[\exp(-rT)(S_T - K)] = 0$$

so that $K = E[S_T]$?

This is *expectation pricing*, which cannot enforce a price.

No-arbitrage approach

The seller of the forward contract can replicate the payoff of the contract at maturity T by borrow S_0 now and buy the commodity.

- When the contract expires, the seller has to pay back the loan of $S_0 e^{rT}$ and deliver the commodity.
- If the seller wrote less than $S_0 e^{rT}$ as the delivery price, then he would lose money with certainty.
- Thus, $S_0 e^{rT}$ is an enforceable price.

Arbitrage opportunity 無風險套利機會

A self-financing trading strategy is requiring no initial investment, having no probability of negative value at expiration, and yet having some possibility of a positive terminal portfolio value.

- Commonly it is assumed that there are no arbitrage opportunities in well functioning and competitive financial markets.

No-arbitrage condition and risk neutral measure

Condition of no arbitrage is equivalent to the existence of an equivalent risk neutral (or martingale) measure.

Equivalent measures

Given two probability measures P and P' defined on the same measurable space $(\Omega, \mathcal{F})$, suppose that

$$P(\omega) > 0 \quad \Longleftrightarrow \quad P'(\omega) > 0, \quad \text{for all } \omega \in \Omega,$$

then P and P' are said to be equivalent measures. In other words, though the two equivalent measures may not agree on the assignment of probability values to individual events, but they always agree as to which events are possible or impossible.

Martingales

Consider a filtered probability space with filtration $\mathbb{F} = \{\mathcal{F}_t; t = 0, 1, \cdots, T\}$. An adapted stochastic process $S = \{S(t); t = 0, 1 \cdots, T\}$ is said to be martingale if it observes

$$E[S(t+s)|\mathcal{F}_t] = S(t) \quad \text{for all } t \geq 0 \text{ and } s \geq 0.$$

- Under the equivalent martingale measure, all discounted price processes of the risky assets are martingales.

Risk neutrality or risk neutral pricing

All assets in the market offer the same rate of return as the riskfree security under this risk neutral measure.

Single period securities models

- The initial prices of M risky securities, denoted by $S_1(0), \cdots, S_M(0)$, are positive scalars that are known at $t = 0$.

- Their values at $t = 1$ are random variables, which are defined with respect to a sample space $\Omega = \{\omega_1, \omega_2, \cdots, \omega_K\}$ of K possible outcomes (or states of the world).

- At $t = 0$, the investors know the list of all possible outcomes, but which outcome does occur is revealed only at the end of the investment period $t = 1$.

- A probability measure P satisfying $P(\omega) > 0$, for all $\omega \in \Omega$, is defined on Ω.

- We use $\boldsymbol{S}$ to denote the price process $\{\boldsymbol{S}(t) : t = 0, 1\}$, where $\boldsymbol{S}(t)$ is the row vector $\boldsymbol{S}(t) = (S_1(t)\ S_2(t) \cdots S_M(t))$.

- The possible values of the asset price process at $t = 1$ are listed in the following $K \times M$ matrix

$$S(1;\Omega) = \begin{pmatrix} S_1(1;\omega_1) & S_2(1;\omega_1) & \cdots & S_M(1;\omega_1) \\ S_1(1;\omega_2) & S_2(1;\omega_2) & \cdots & S_M(1;\omega_2) \\ \cdots & \cdots & \cdots & \cdots \\ S_1(1;\omega_K) & S_2(1;\omega_K) & \cdots & S_M(1;\omega_K) \end{pmatrix}.$$

- Since the assets are limited liability securities, the entries in $S(1;\Omega)$ are non-negative scalars.

- We also assume the existence of a strictly positive riskless security or bank account, whose value is denoted by S_0. Without loss of generality, we take $S_0(0) = 1$ and the value at time 1 to be $S_0(1) = 1 + r$, where $r \geq 0$ is the deterministic interest rate over one period.

- We define the discounted price process by

$$S^*(t) = S(t)/S_0(t), \quad t = 0, 1,$$

that is, we use the riskless security as the *numeraire* or *accounting unit*.

- The payoff matrix of the discounted price processes of the M risky assets and the riskless security can be expressed in the form

$$\widehat{S}^*(1;\Omega) = \begin{pmatrix} 1 & S_1^*(1;\omega_1) & \cdots & S_M^*(1;\omega_1) \\ 1 & S_1^*(1;\omega_2) & \cdots & S_M^*(1;\omega_2) \\ \cdots & \cdots & \cdots & \cdots \\ 1 & S_1^*(1;\omega_K) & \cdots & S_M^*(1;\omega_K) \end{pmatrix}.$$

- An investor adopts a *trading strategy* by selecting a portfolio of the M assets at time 0. The number of units of asset m held in the portfolio from $t = 0$ to $t = 1$ is denoted by $h_m, m = 0, 1, \cdots, M$. The scalars h_m can be positive (long holding), negative (short selling) or zero (no holding).

- Let $V = \{V_t : t = 0, 1\}$ denote the value process that represents the total value of the portfolio over time. It is seen that

$$V_t = h_0 S_0(t) + \sum_{m=1}^{M} h_m S_m(t), \quad t = 0, 1.$$

- Let G be the random variable that denotes the total gain generated by investing in the portfolio. We then have

$$G = h_0 r + \sum_{m=1}^{M} h_m \Delta S_m.$$

- If there is no withdrawal or addition of funds within the investment horizon, then

$$V_1 = V_0 + G.$$

- Suppose we use the bank account as the numeraire, and define the discounted value process by $V_t^* = V_t / S_0(t)$ and discounted gain by $G^* = V_1^* - V_0^*$, we then have

$$\begin{aligned} V_t^* &= h_0 + \sum_{m=1}^{M} h_m S_m^*(t), \quad t = 0, 1; \\ G^* &= V_1^* - V_0^* = \sum_{m=1}^{M} h_m \Delta S_m^*. \end{aligned}$$

Asset span

- Consider two risky securities whose discounted payoff vectors are

$$\boldsymbol{S}_1^*(1) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \text{and} \quad \boldsymbol{S}_2^*(1) = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}.$$

- The payoff vectors are used to form the payoff matrix

$$S^*(1) = \begin{pmatrix} 1 & 3 \\ 2 & 1 \\ 3 & 2 \end{pmatrix}.$$

- Let the current discounted prices be represented by the row vector $\boldsymbol{S}^*(0) = (1 \quad 2)$.

- We write $\boldsymbol{h}$ as the column vector whose entries are the weights of the securities in the portfolio. The current portfolio value and the discounted portfolio payoff are given by $\boldsymbol{S}^*(0)\boldsymbol{h}$ and $S^*(1)\boldsymbol{h}$, respectively.

- The set of all portfolio payoffs via different holding of securities is called the *asset span* $\mathcal{S}$. The asset span is seen to be the column space of the payoff matrix $S^*(1)$.

- The asset span consists of all vectors of the form $h_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + h_2 \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$, where h_1 and h_2 are scalars.

- If an added security lies inside $\mathcal{S}$, then its payoff can be expressed as a linear combination of $\boldsymbol{S}_1^*(1)$ and $\boldsymbol{S}_2^*(1)$. In this case, it is said to be a *redundant security*.

- A securities model is said to be *complete* if every payoff vector lies inside the asset span. This occurs if and only if the dimension of the asset span equals the number of possible states.

Law of one price

1. The law of one price states that all portfolios with the same payoff have the same price.

2. Consider two portfolios with different portfolio weights $\boldsymbol{h}$ and $\boldsymbol{h}'$. Suppose these two portfolios have the same discounted payoff, that is, $S^*(1)\boldsymbol{h} = S^*(1)\boldsymbol{h}'$, then the law of one price infers that $\boldsymbol{S}^*(0)\boldsymbol{h} = \boldsymbol{S}^*(0)\boldsymbol{h}'$.

3. A necessary and sufficient condition for the law of one price to hold is that a portfolio with zero payoff must have zero price.

4. If the law of one price fails, then it is possible to have two trading strategies $\boldsymbol{h}$ and $\boldsymbol{h}'$ such that $S^*(1)\boldsymbol{h} = S^*(1)\boldsymbol{h}'$ but $\boldsymbol{S}^*(0)\boldsymbol{h} > \boldsymbol{S}^*(0)\boldsymbol{h}'$.

Pricing functional

- Given a discounted portfolio payoff $\boldsymbol{x}$ that lies inside the asset span, the payoff can be generated by some linear combination of the securities in the securities model. We have $\boldsymbol{x} = S^*(1)\boldsymbol{h}$ for some $\boldsymbol{h} \in \mathbb{R}^M$.

- The current value of the portfolio is $\boldsymbol{S}^*(0)\boldsymbol{h}$, where $\boldsymbol{S}^*(0)$ is the price vector.

- We may consider $\boldsymbol{S}^*(0)\boldsymbol{h}$ as a pricing functional $F(\boldsymbol{x})$ on the payoff $\boldsymbol{x}$. If the law of one price holds, then the pricing functional is single-valued. Furthermore, it is a linear functional, that is,

$$F(\alpha_1\boldsymbol{x}_1 + \alpha_2\boldsymbol{x}_2) = \alpha_1 F(\boldsymbol{x}_1) + \alpha_2 F(\boldsymbol{x}_2)$$

  for any scalars α_1 and α_2 and payoffs $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$.

Arrow security and state price

- Let $\boldsymbol{e}_k$ denote the k^{th} coordinate vector in the vector space $\mathbb{R}^K$, where $\boldsymbol{e}_k$ assumes the value 1 in the k^{th} entry and zero in all other entries. The vector $\boldsymbol{e}_k$ can be considered as the discounted payoff vector of a security, and it is called the Arrow security of state k.

- Suppose the securities model is complete and the law of one price holds, then the pricing functional F assigns unique value to each Arrow security. We write $s_k = F(\boldsymbol{e}_k)$, which is called the state price of state k.

Arbitrage opportunities and risk neutral probability measure

- An *arbitrage opportunity* is some trading strategy that has the following properties: (i) $V_0^* = 0$, (ii) $V_1^*(\omega) \geq 0$ and $EV_1^*(\omega) > 0$, where E is the expectation under the actual probability measure P.

- In financial markets with no arbitrage opportunities, every investor should use such risk neutral probability measure (though not necessarily unique) to find the fair value of a portfolio, irrespective to the risk preference of the investor.

A probability measure Q on Ω is a risk neutral probability measure if it satisfies

(i) $Q(\omega) > 0$ for all $\omega \in \Omega$, and

(ii) $E_Q[\Delta S_m^*] = 0, m = 1, \cdots, M$, where E_Q denotes the expectation under Q.

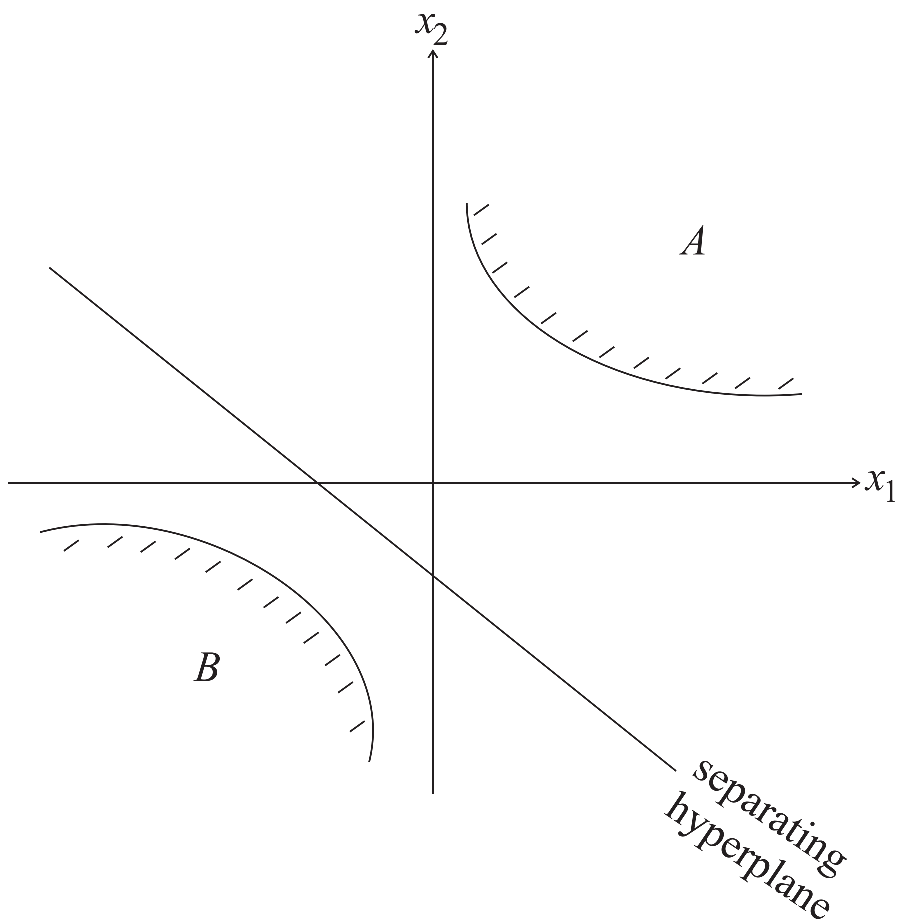
Note that $E_Q[\Delta S_m^*] = 0$ is equivalent to $S_m^*(0) = \sum_{k=1}^{K} Q(\omega_k) S_m^*(1; \omega_k)$.

Fundamental Theorem of Asset Pricing

No arbitrage opportunities exist if and only if there exists a risk neutral probability measure Q.

The proof of the Theorem requires the Separating Hyperplane Theorem.

The *Separating Hyperplane Theorem* states that if A and B are two non-empty disjoint convex sets in a vector space V, then they can be separated by a hyperplane.



The hyperplane (represented by a line in $\mathbb{R}^2$) separates the two convex sets A and B in $\mathbb{R}^2$.

The hyperplane $[\boldsymbol{f}, \alpha]$ separates the sets A and B in $\mathbb{R}^n$ if there exists α such that $\boldsymbol{f} \cdot \boldsymbol{x} \geq \alpha$ for all $\boldsymbol{x} \in A$ and $\boldsymbol{f} \cdot \boldsymbol{y} < \alpha$ for all $\boldsymbol{y} \in B$.

For example, the hyperplane $\left[\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, 0\right]$ separates the two disjoint convex sets $A = \left\{ \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} : x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \right\}$

and $B = \left\{ \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} : x_1 < 0, x_2 < 0, x_3 < 0 \right\}$ in $\mathbb{R}^3$.

Proof of Theorem

"$\Leftarrow$ part".

Assume a risk neutral probability measure Q exists, that is, $\widehat{\boldsymbol{S}}^*(0) = \boldsymbol{\pi}\widehat{S}^*(1;\Omega)$, where $\boldsymbol{\pi} = (Q(\omega_1) \cdots Q(\omega_K))$. Consider a trading strategy $\boldsymbol{h} = (h_1 \quad \cdots \quad h_M)^T \in \mathbb{R}^M$ such that $S^*(1;\Omega)\boldsymbol{h} \geq 0$ in all $\omega \in \Omega$ and with strict inequality in some states.

Now consider $\widehat{\boldsymbol{S}}^*(0)\boldsymbol{h} = \boldsymbol{\pi}\widehat{S}^*(1;\Omega)\boldsymbol{h}$, it is seen that $\widehat{\boldsymbol{S}}^*(0)\boldsymbol{h} > 0$ since all entries in $\boldsymbol{\pi}$ are strictly positive and entries in $\widehat{S}^*(1;\Omega)\boldsymbol{h}$ are either zero or strictly positive. Hence, no arbitrage opportunities exist.

"$\Rightarrow$ part".

First, we define the subset U in $\mathbb{R}^{K+1}$ which consists of vectors of the form $\begin{pmatrix} -\widehat{\boldsymbol{S}}^*(0)\boldsymbol{h} \\ \widehat{\boldsymbol{S}}^*(1;\omega_1)\boldsymbol{h} \\ \vdots \\ \widehat{\boldsymbol{S}}^*(1;\omega_K)\boldsymbol{h} \end{pmatrix}$, where $\widehat{\boldsymbol{S}}^*(1;\omega_k)$ is the k^{th} row in $\widehat{S}^*(1;\Omega)$ and $\boldsymbol{h} \in \mathbb{R}^M$ represents a trading strategy. This subset is seen to be a convex subspace.

Consider another subset $\mathbb{R}_+^{K+1}$ defined by

$$\mathbb{R}_+^{K+1} = \{\boldsymbol{x} = (x_0\ x_1 \cdots x_K)^T \in \mathbb{R}^{K+1} : x_i \geq 0 \quad \text{for all} \quad 0 \leq i \leq K\},$$

which is a convex set in $\mathbb{R}^{K+1}$.

We claim that the non-existence of arbitrage opportunities implies that U and $\mathbb{R}_+^{K+1}$ can only have the zero vector in common.

Assume the contrary, suppose there exists a non-zero vector $\boldsymbol{x} \in U \cap \mathbb{R}_+^{K+1}$. Since there is a trading strategy vector $\boldsymbol{h}$ assoicated with every vector in U, it suffices to show that the trading strategy $\boldsymbol{h}$ associated with $\boldsymbol{x}$ always represents an arbitrage opportunity.

We consider the following two cases:

(i) $-\widehat{\boldsymbol{S}}^*(0)\boldsymbol{h} = 0$ or $-\widehat{\boldsymbol{S}}^*(0)\boldsymbol{h} > 0$. When $\widehat{\boldsymbol{S}}^*(0)\boldsymbol{h} = 0$, since $\boldsymbol{x} \neq \boldsymbol{0}$ and $\boldsymbol{x} \in R_+^{K+1}$, then the entries $\widehat{\boldsymbol{S}}(1;\omega_k)\boldsymbol{h}, k = 1, 2, \cdots K$, must be all greater than or equal to zero, with at least one strict inequality. In this case, $\boldsymbol{h}$ is seen to represent an arbitrage opportunity.

(ii) When $\widehat{\boldsymbol{S}}^*(0)\boldsymbol{h} < 0$, all the entries $\widehat{\boldsymbol{S}}(1;\omega_k)\boldsymbol{h}, k = 1, 2, \cdots, K$ must be all non-negative. Correspondingly, $\boldsymbol{h}$ represents a dominant trading strategy and in turns $\boldsymbol{h}$ is an arbitrage opportunity.

Since $U \cap R_{+}^{K+1} = \{\mathbf{0}\}$, by the Separating Hyperplane Theorem, there exists a hyperplane that separates $\mathbb{R}_{+}^{K+1} \backslash \{\mathbf{0}\}$ and U. Let $\boldsymbol{f} \in \mathbb{R}^{K+1}$ be the normal to this hyperplane, then we have $\boldsymbol{f} \cdot \boldsymbol{x} > \boldsymbol{f} \cdot \boldsymbol{y}$, where $\boldsymbol{x} \in \mathbb{R}_{+}^{K+1} \backslash \{\mathbf{0}\}$ and $\boldsymbol{y} \in U$.

[*Remark*: We may have $\boldsymbol{f} \cdot \boldsymbol{x} < \boldsymbol{f} \cdot \boldsymbol{y}$, depending on the orientation of the normal. However, the final conclusion remains unchanged.]

Since U is a linear subspace so that a negative multiple of $\boldsymbol{y} \in U$ also belongs to U, the condition $\boldsymbol{f} \cdot \boldsymbol{x} > \boldsymbol{f} \cdot \boldsymbol{y}$ holds only if $\boldsymbol{f} \cdot \boldsymbol{y} = 0$ for all $\boldsymbol{y} \in U$.

We have $\boldsymbol{f} \cdot \boldsymbol{x} > 0$ for all $\boldsymbol{x}$ in $\mathbb{R}_+^{K+1} \backslash \{\mathbf{0}\}$. This requires all entries in $\boldsymbol{f}$ to be strictly positive. From $\boldsymbol{f} \cdot \boldsymbol{y} = 0$, we have

$$-f_0 \widehat{\boldsymbol{S}}^*(0)\boldsymbol{h} + \sum_{k=1}^{K} f_k \widehat{\boldsymbol{S}}^*(1;\omega_k)\boldsymbol{h} = 0$$

for all $\boldsymbol{h} \in \mathbb{R}^M$, where $f_j, j = 0, 1, \cdots, K$ are the entries of $\boldsymbol{f}$. We then deduce that

$$\widehat{\boldsymbol{S}}^*(0) = \sum_{k=1}^{K} Q(\omega_k)\widehat{\boldsymbol{S}}^*(1;\omega_k), \text{ where } Q(\omega_k) = f_k/f_0.$$

Consider the first component in the vectors on both sides of the above equation. They both correspond to the current price and discounted payoff of the riskless security, and all are equal to one. We then obtain

$$1 = \sum_{k=1}^{K} Q(\omega_k).$$

We obtain the risk neutral probabilities $Q(\omega_k), k = 1, \cdots, K$, whose sum is equal to one and they are all strictly positive since $f_j > 0, j = 0, 1, \cdots, K$.

Remark

Corresponding to each risky asset,

$$S_m^*(0) = \sum_{k=1}^{K} Q(\omega_k) S_m^*(1; \omega_k), \quad m = 1, 2, \cdots, M.$$

Hence, the current price of any risky security is given by the expectation of the discounted payoff under the risk neutral measure Q.