Recent Development of Modern Control Theory and its Application

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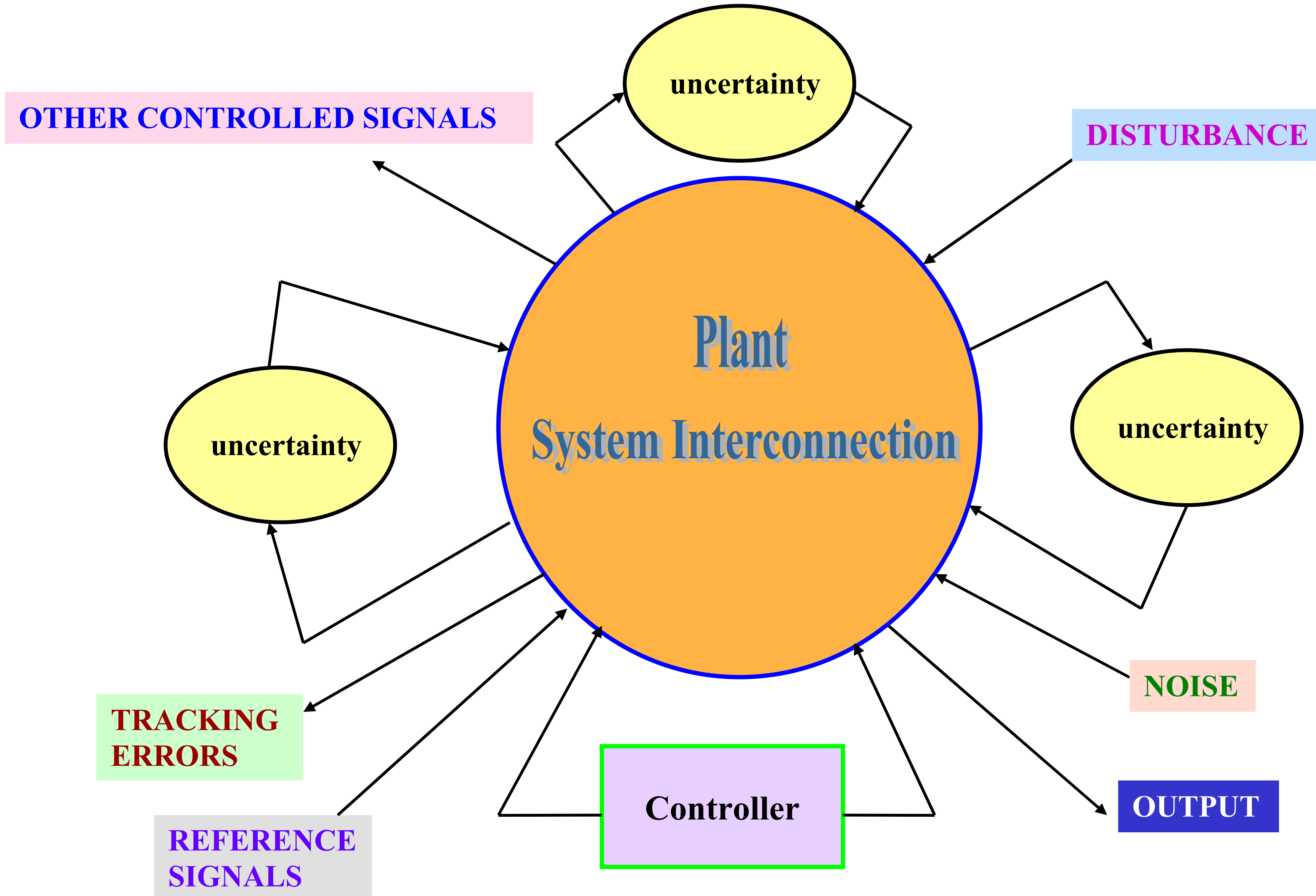
Outline

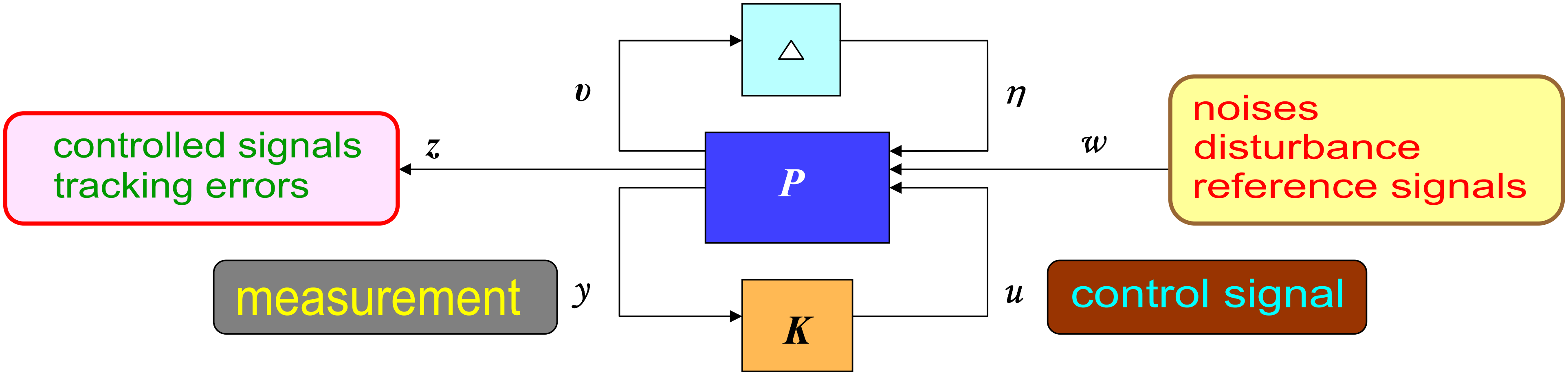
- **Brief History of Control and Filtering design**
- **Nonlinear descriptor estimator design**
- **Application to sensor fault diagnosis**
- **Robust stability theory for nonlinear singular systems**
- **Computation of the (J, J')-Lossless Factorization**



General System Interconnection

Brief History on Control (Filtering) System Design

- **Linear Control/Filter (60's - 70's)(PID control, LQR design, LQG design, Adaptive design, Kalman filter, ...)**
- **Robust Control/Filter (80's) (H_∞ Control, Small Gain Theorem,...)**
- **Nonlinear Control/Filter (90's -) (Non-linear H_∞, Sliding mode, Non-linear adaptive, ...)**
- **Intelligent Control/Filter (90's -) (Neural Networks, Fuzzy, Genetic Algorithm, ...)**
- **Industry Applications : aerospace, steel, ship, car, chemical plant, power plant,...**
- **Design Systems: dynamical systems, singular systems, delay systems, uncertain systems, stochastic systems, large scale systems, chaotic systems,...**
- **Reliable Numerical software development: Simulink, LMI,...**



General Linear Fractional Transformation (LFT) Framework

$$\begin{bmatrix} v \\ z \\ y \end{bmatrix} = P \begin{bmatrix} \eta \\ w \\ u \end{bmatrix}$$

$$\eta = \triangle v$$

$$u = Ky$$

$$\min \|T_{zw}\|_{\infty} = \min \left(\sup_w \bar{\sigma}\,(T_{zw}(jw))\right)$$

Current and Future trend

- **New problems**: fault detection, fault tolerance, missing signals, disturbance, noise in measurement, random impulse, unknown delays, reliability, ...
- **New systems**: network control systems, control/filter design in bio-informatic systems, data-mining in communication systems/business systems,...
- **New challenge**: new control concept, new mathematical analysis, new learning techniques, new robust numerical tools, new powerful software, new optimization tools, new statistical tools, new clustering designs,...

Team effort is required from experts of different research areas:

- Control and Automation Engineers,
- Applied Mathematicians,
- Computer Scientists,
- Electronic System Engineers,
- Biologists...

My first Mathematical Control problem:

Output feedback Pole Assignment Problem (1982 -1986)

Given a linear system

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \tag{1}$$

Output feedback design: $u = Ky$

The problem is to find a suitable feedback gain matrix K**, such that** $\dot{x} = (A + BKC)x$ **is stable with specific performance.**

That problem is equivalent to find a suitable K **such that** $A + BKC$ **has a specific set of poles (eigenvalues) with -ve real part.(Eigenvalue assignment).**

Later development:

To develop reliable numerical software for matlab control tool box... (many papers are published).

Eigenstructure assignment

Robust pole assignmne under perturbation.(1986 - ...)

Approximate pole assignment. (1986 - 2000)

Regional pole assignment for large scale system...

Pole assignment for singular system (E. Chu(1986), L R Fletcher(1988), D.Chu & D.Ho(2002)...)

$$\begin{cases} E\dot{x} = Ax + Bu \\ y = Cx \end{cases} \tag{2}$$

State/noise estimator for descriptor systems with application to sensor fault diagnosis

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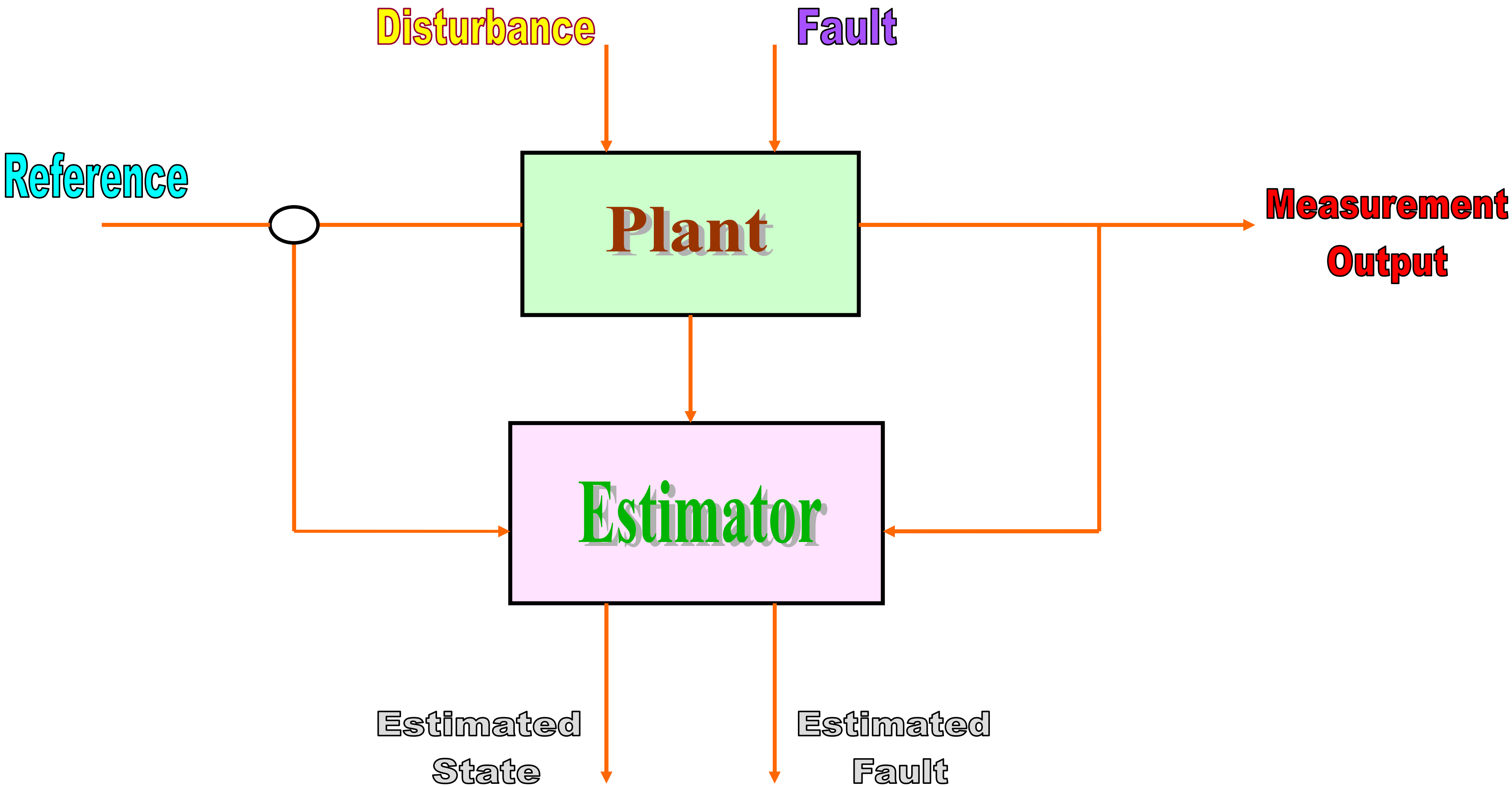


Estimator Design

PP

Introduction

- **Descriptor system is also referred to as singular system, generalized state-space system, implicit system, semi-state system, algebraic differential equation system, etc.**

- **Descriptor system model arises from a convenient and natural modelling process, and has a profound background in engineering practise, social science, network analysis, etc.**

- **Descriptor system model can characterize a more general class of systems than a normal system model. Great efforts were made to investigate descriptor system theory and applications during the past thirty years; see [1, 2, 3] and the references therein.**

[1] E. Brenan, S. L. Campbell, and L. R. Petzold, "*Numerical Solution of Initial-Value Problems in Differential-Algebraic Equations***," SIAM, Philadelphia, 1996.**

[2] L. Dai, "*Singular Control Systems***, "Springer-Verlag, New York, 1989.**

[3] D. W. C. Ho, and Z. Gao, "Bezout identity related to reduced-order observer-based controllers for singular systems," *Automatica***, vol.37, no.10, pp. 1655-1662, 2001.**

Background and Preliminaries

To illustrate the observer design approach in the literature. Consider

$$\begin{cases} E\dot{x} = Ax + Bu + d \\ y = Cx + d_n \end{cases} \quad (3)$$

where

$x \in \mathcal{R}^n$ **is the descriptor vector,**

$u \in \mathcal{R}^m$ **and** $y \in \mathcal{R}^p$ **are respectively the control input and measure output vectors,**

$d \in \mathcal{R}^n$ **is disturbance vector;** $d_n \in \mathcal{R}^p$ **is the measurement noise;**

E, $A \in \mathcal{R}^{n\times n}$, E **may be singular, when** $E = I$, **the system degenerates to a normal system.**

B **and** C **are constant real matrices of appropriate dimensions;**

$Cx + d_n \neq 0$ **is tolerated, otherwise there is a total failure of the sensor.**

It is known that the state of the descriptor system is completely determined for all $t >= 0$ by $x(0^-)$ and u. However, due to the presence of infinite frequency natural modes, x could display an impulsive behavior. In particular, x could have generalized impulses, whenever u or $\dot{u}$ are discontinuous. This has the adverse effects of limiting the class of acceptable input functions and, making the system highly susceptible to noise.

In the 80s'

Solutions:

(i) use state feedback control to modify the system structure so as to smooth the state trajectories.

(ii) use state feedback to replace the infinite frequency modes by finite frequency modes.

(iii) to estimate the states by observer design ...

+ many other objectives...

New estimator Design Objective:

(i) To develop a new descriptor dynamical system

(ii) To find an asymptotic state estimator $\hat{x}$ such that $e = x - \hat{x}$ as small as possible throughout the process, subject to impulse-free, finite observable, stable....

- **The pair (E, A) is regular provided that**

$$\det(sE - A) \not\equiv 0, \ \ s \in \mathcal{C}, \tag{4}$$

which ensures the plant $E\dot{x}(t) = Ax(t)$ is solvable, i.e. this plant possesses a unique solution for any given consistent initial value.

- **The pair (E, A) is internally stable provided that**

$$\text{rank}(sE - A) = n, \ \ \forall s \in \mathcal{C}_+. \tag{5}$$

- **The pair (E, A) is internally proper (also called impulse-free or causal) provided that**

$$\text{rank}\begin{bmatrix} E & 0 \\ A & E \end{bmatrix} = n + \text{rank}(E), \tag{6}$$

or equivalently

$$\deg[\det(sE - A)] = \text{rank}(E). \tag{7}$$

Clearly, the pair (E, A) must be regular if the plant is internally proper.

- **The triple (E, A, C) is finite detectable provided that**

$$\text{rank}\begin{bmatrix} sE - A \\ C \end{bmatrix} = n, \forall\ s \in \mathcal{C}_+. \tag{8}$$

There exists a matrix K such that $(E, A - KC)$ is internally stable if and only if the plant (E, A, C) is finite detectable.

- **The triple (E, A, C) is impulsive observable provided that**

$$\text{rank}\begin{bmatrix} E & 0 \\ A & E \\ C & 0 \end{bmatrix} = n + \text{rank}(E). \tag{9}$$

There exists a matrix K such that $(E, A - KC)$ is internally proper if and only if the plant (E, A, C) is impulsive observable. Furthermore, there exists a matrix K such that $(E, A - KC)$ is internally proper and stable if and only if the plant (E, A, C) is impulsive observable and finite detectable.

- **The pair (E, A) is regular, internally proper and stable, if and only if there exists a matrix X such that**

$$E^T X = X^T E \geq 0, \quad A^T X + X^T A < 0. \tag{10}$$

To illustrate the motivation of our work, let us comment on the PI observer design approach in the literature.

Let $\hat{x}$ is the estimated vector,

$\hat{f}$ is a vector representing the integral of the weighted output estimation error,

L_P^0 and L_P are the proportional gains,

L_I is the integral gain.

Let $e = z - T_1 Ex$, the estimation error dynamics is governed by

$$\begin{bmatrix} \dot{e} \\ \dot{\hat{f}} \end{bmatrix} = \begin{bmatrix} F & L_I^0 \\ -L_I C & 0 \end{bmatrix} \begin{bmatrix} e \\ \hat{f} \end{bmatrix} + \begin{bmatrix} L_P^0 + L_P \\ L_I(I - CT_2) \end{bmatrix} d \tag{11}$$

where $F = T_1 A - L_P C$, $L_P^0 = FT_2$, and the gain matrices

L_P, L_I^0 and L_I are selected such that the error dynamics is stable.

Clearly, the noise d in the error equation will be amplified unavoidably if the gains L_P and L_I are high.

Therefore, the PI observer in (Niemann *et al* 1995) also cannot obtain satisfactory tracking performance for the plant with measurement noises.

Example: Consider the following descriptor system

$$\begin{cases} E\dot{x} = Ax + Bu \\ y = Cx + d \end{cases} \tag{12}$$

where $E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $A = \begin{bmatrix} 2 & -1 \\ -1 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $u = \begin{bmatrix} sin(t) \\ cos(t) \end{bmatrix}$, $d = \begin{bmatrix} 0.5 \\ 2 \end{bmatrix}$.

PI observer

Choose $T_1 = \begin{bmatrix} 0 & 0 \\ 0.5 & 0 \end{bmatrix}$ **and** $T_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$ $L_I^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. **Design** $L_P = \begin{bmatrix} 2 & 0 \\ 1 & 19.5 \end{bmatrix}$ **and** $L_I = \begin{bmatrix} 2 & 0 \\ 0 & 64 \end{bmatrix}$ **such that the poles of the observer are located inside the set** $\{-4,\ -16,\ -1+i,\ -1-i\}$. **Then we can compute that** $L_P^0 = \begin{bmatrix} -2 & 0 \\ 0 & -10 \end{bmatrix}$, $F = \begin{bmatrix} -2 & 0 \\ 0 & -20 \end{bmatrix}$ **and** $T_1B = \begin{bmatrix} 0 & 0 \\ 0.5 & 0.5 \end{bmatrix}$.

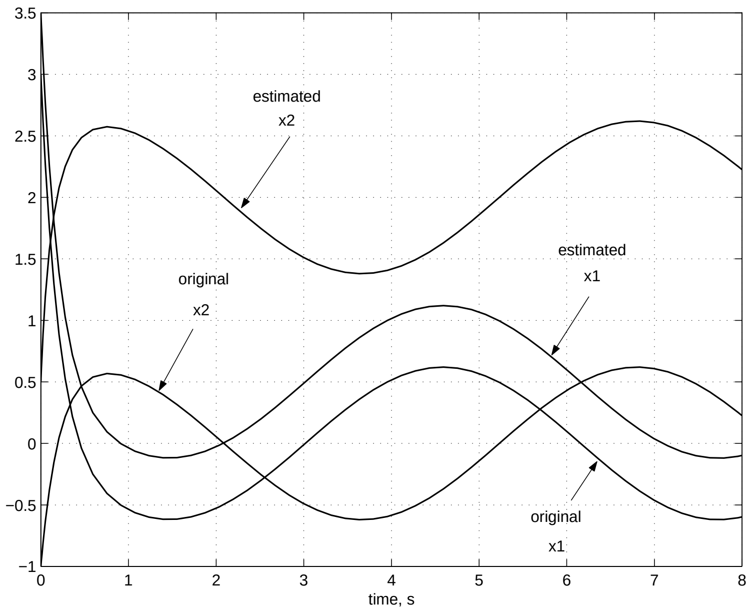


Figure 1: State estimation via PI observer

Figure 1 shows this kind PI observer cannot obtain satisfactory estimation.

- **In the work [14], a proportional multiple-integral (PMI) estimator was proposed to simultaneously estimate the descriptor system state and a class of slow-varying polynomial output disturbances.**

$$\begin{cases} E\dot{x} = Ax + Bu + B_d d \\ \dot{x}_0 = Cx + d \\ y_0 = x_0. \end{cases} \tag{13}$$

where $x_0 = \int_0^t y(\tau)d\tau$ **and** $B_d \in \mathcal{R}^{n\times p}$ **is a known input disturbance matrix (may be a low-rank matrix and may even be zero).**

- $d(t) \in \mathcal{R}^p$ **is the measurement output noise and** $B_d d(t)$ **in the dynamic equation characterizes the unknown input uncertainty or modelling error.**

[14] Z. Gao, and D. W. C. Ho, "Proportional multiple-integral observer design for descriptor systems with measurement output disturbances," *IEE Proc. - Control Theory Appl.*, **vol.151, no.3, pp.279-288, 2004.**

Let

$$z = \begin{bmatrix} x \\ x_0 \end{bmatrix}, \ \bar{E} = \begin{bmatrix} E & 0 \\ 0 & I \end{bmatrix}, \ \bar{A} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}, \tag{14}$$
$$\bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \ \bar{M} = \begin{bmatrix} B_d \\ I \end{bmatrix}, \ \bar{C}_0 = \begin{bmatrix} 0 & I \end{bmatrix},$$

then the augmented plant (13) can be rewritten as follows:

$$\begin{cases} \bar{E}\dot{z} = \bar{A}z + \bar{B}u + \bar{M}d \\ y_0 = \bar{C}_0 z. \end{cases} \tag{15}$$

$$d = A_0 + A_1 t + A_2 t^2 + \cdots + A_{q-1} t^{q-1} \tag{16}$$

where A_i $(i = 0, 1, 2, \cdots, q-1)$ are the constant matrices, the values of which are unknown. Clearly, the q^{th} derivative of this disturbance is zero, i.e., $d^{(q)} = 0$, then we can call q as the index of disturbance. Clearly, the disturbance in the form (16) may be unbounded, which is more general than the constant disturbance considered in (Koenig and Mammar 2002).

The observer can be designed as:

$$\begin{cases} \bar{E}\dot{\hat{z}} = \bar{A}\hat{z} + \bar{B}u + L_I(y_0 - \bar{C}_0\hat{z}) + \bar{M}\hat{f}_q \\ \dot{\hat{f}}_q = L_I^q(y_0 - \bar{C}_0\hat{z}) + \hat{f}_{q-1} \\ \dot{\hat{f}}_{q-1} = L_I^{q-1}(y_0 - \bar{C}_0\hat{z}) + \hat{f}_{q-2} \\ \vdots \\ \dot{\hat{f}}_2 = L_I^2(y_0 - \bar{C}_0\hat{z}) + \hat{f}_1 \\ \dot{\hat{f}}_1 = L_I^1(y_0 - \bar{C}_0\hat{z}). \end{cases} \tag{17}$$

Here, $\hat{z} \in \mathcal{R}^{n+p}$ is an estimation of the descriptor state vector z, and $\hat{f}_i \in \mathcal{R}^p$ $(i = 1, 2, \ldots, q)$ is an estimation of the $(q-i)^{th}$ derivative of the disturbance $d(t)$ in the form (16), $L_I \in \mathcal{R}^{(n+p)\times p}$ and $L_I^i \in \mathcal{R}^{p\times p}$, $i = 1, 2, \ldots, q$ are integral gains. Now we have the following statement.

Theorem 1 ***For the plant (15), there exists an asymptotic descriptor observer in the form (17) if the triple*** (E, A, C) ***is finite observable, and*** $\operatorname{rank}\begin{bmatrix} A & B_d \\ C & I \end{bmatrix} = n + p.$

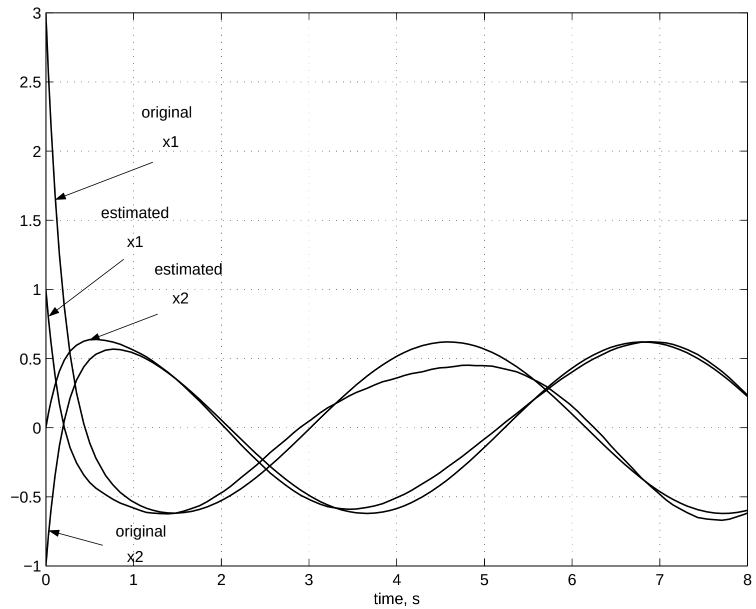


Figure 2: State estimation via new observer with $q = 1$: $\dot{d}$ bounded

$$d = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0.5 + 0.2sin(t) + 0.4sin(50t) \\ 0.25t + 2 + sin(2t) + 0.8sin(20t) \end{bmatrix}, \tag{18}$$

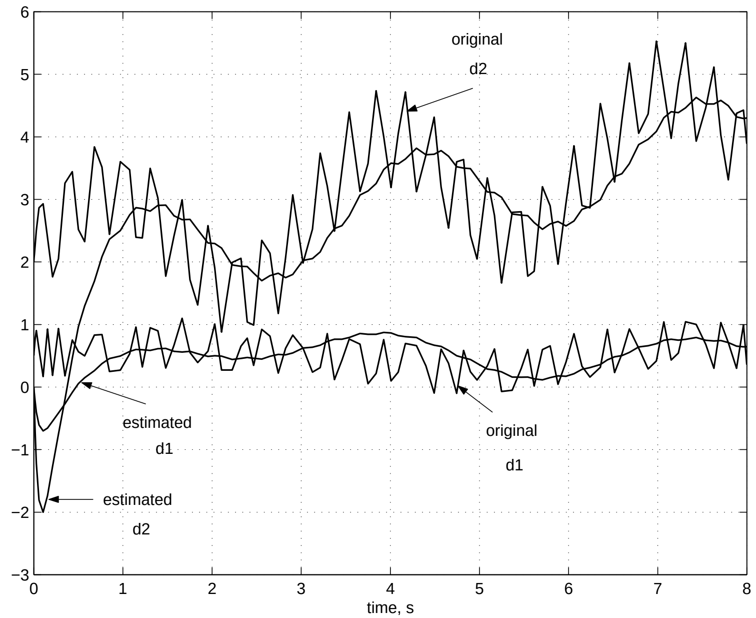


Figure 3: Disturbance estimation via new observer with $q = 1$: $\dot{d}$ bounded

- **However, when the output disturbance is a high-frequency signal, one cannot obtain satisfactory disturbance estimation by using the PMI technique given in [14].**

New Descriptor state/disturbance estimator design

- **Let**

$$\bar{x} = \begin{bmatrix} x \\ d \end{bmatrix}, \quad \bar{E} = \begin{bmatrix} E & 0 \\ 0 & 0_{p\times p} \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} A & 0 \\ 0 & -I_p \end{bmatrix},$$
$$\bar{B} = \begin{bmatrix} B \\ 0_{p\times m} \end{bmatrix}, \quad \bar{N} = \begin{bmatrix} B_d \\ I_p \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} C & I_p \end{bmatrix}, \tag{19}$$

and we can construct the following augmented plant

$$\begin{cases} \bar{E}\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u + \bar{N}d \\ y = \bar{C}\bar{x}. \end{cases} \tag{20}$$

- **The descriptor state** x **and the disturbance** d **are both the descriptor state vectors of the augmented plant. Hence, if an asymptotic state estimator can be constructed for the augmented plant (20), then this estimator is a** *simultaneous state and disturbance estimator* **for the original plant (13)**

- **Now we develop a new descriptor estimator design approach for the augmented plant (20). Consider the descriptor dynamical system as follows**

$$\bar{E}\dot{\hat{\bar{x}}} = \bar{A}\hat{\bar{x}} + \bar{K}_p(y - \bar{C}\hat{\bar{x}}) + \bar{B}u + \bar{N}\hat{d}, \tag{21}$$

where $\hat{\bar{x}} \in \mathcal{R}^{n+p}$ **is an estimation of the descriptor state vector** $\bar{x}$**;** $\hat{d} \in \mathcal{R}^p$ **is an estimation of the disturbance** d**;** $\bar{K}_p \in \mathcal{R}^{(n+p)\times p}$ **is the gain matrix to be designed. Now we have the following statement.**

- **Theorem 2** ***For the plant (20), there exists an asymptotic estimator in the form of (21) if and only if the pair*** $(E, A - B_d C)$ ***is internally proper and stable.***

  Proof: See Appendix A

Nonlinear descriptor estimator design

- the estimator design issues becomes more difficult and challenging since the existence and convergence properties have to be considered simultaneously in the design process.
- In fact, only limited work on nonlinear descriptor system is available in the literature [15, 16, 17].
- To the best of our knowledge, the study of estimator design for nonlinear descriptor systems with output disturbances has not been reported in the literature. The problem remains important and unsolved, and there is a strong incentive to improve the estimator design for nonlinear descriptor systems.

[15] S. Kaprielian, and J. Turi, "An observer for a nonlinear descriptor system," *Proc. IEEE Conf. Deci. Control*, pp. 975-976, 1992.

[16] G. Lu, D. W. C. Ho, and Y. Zhang, "Observers for a class of descriptor systems with Lipschitz constraint," *Proc. the American Control Conf.*, pp.3474-3479, 2004.

[17] D. N. Shields, "Observer design and detection for nonlinear de-

- **When an additional nonlinear term appears, the plant (13) becomes**

$$\begin{cases} E\dot{x} = Ax + Bu + B_d d + f(t, x, u) \\ y = Cx + d, \end{cases} \tag{22}$$

where $f(t, x, u) \in \mathcal{R}^n$ **is a real nonlinear vector function.**

- **Let**

$$\bar{f}(t, x, u) = \begin{bmatrix} f(t, x, u) \\ 0_{p \times n} \end{bmatrix}, \tag{23}$$

and by using $\bar{E}$, $\bar{A}$ **and so on as described by (19), we can construct the following augmented descriptor nonlinear system:**

$$\begin{cases} \bar{E}\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u + \bar{N}d + \bar{f}(t, x, u) \\ y = \bar{C}\bar{x}. \end{cases} \tag{24}$$

- **Consider the descriptor nonlinear dynamical system as follows**

$$\bar{E}\dot{\hat{\bar{x}}} = (\bar{A} + \bar{N}\bar{C}_d - \bar{K}_p\bar{C})\hat{\bar{x}} + \bar{B}u + \bar{K}_p y + \bar{f}(t, \hat{x}, u), \tag{25}$$

where $\hat{\bar{x}} = \begin{bmatrix} \hat{x} \\ \hat{d} \end{bmatrix} \in \mathcal{R}^{n+p}$ **is the estimation of** $\bar{x} = \begin{bmatrix} x \\ d \end{bmatrix} \in \mathcal{R}^{n+p}$**, and** $\bar{K}_p \in \mathcal{R}^{(n+p)\times p}$ **is the estimator gain matrix to be designed. Now we have the following statement.**

- **Theorem 3** ***For the plant (24), there exists an asymptotic descriptor estimator in the form of (25) if***
(i) the nonlinear function $f(t, x, u)$ ***is Lipschitz with a Lipschitz constant*** γ***, i.e.***

$$\|f(t,x,u) - f(t,\hat{x},u)\| \leq \gamma\|x - \hat{x}\|, \ \forall \ (t,x,u),(t,\hat{x},u) \in \mathcal{R}\times\mathcal{R}^n\times\mathcal{R}^m; \tag{26}$$

(ii) there exist matrices $\bar{X}$ ***and*** $\bar{K}_p$ ***such that***

$$\bar{E}^T\bar{X} = \bar{X}^T\bar{E} \geq 0, \tag{27}$$

$$\begin{bmatrix} (\bar{A} + \bar{N}\bar{C}_d - \bar{K}_p\bar{C})^T\bar{X} + \bar{X}^T(\bar{A} + \bar{N}\bar{C}_d - \bar{K}_p\bar{C}) + \gamma^2 I & \bar{X}^T \\ \bar{X} & -I \end{bmatrix} < 0. \tag{28}$$

Proof. See Appendix D.

Application to sensor fault diagnosis

- **In this section, we will apply the proposed estimator technique to the sensor fault diagnosis. We will consider the following three cases.**
- **In case 1, the input nonlinear term and the sensor fault both exist in the plant.**
- **In case 2, the input nonlinear term, the sensor fault and the output noise appear in the plant at the same time.**
- **In case 3, the nonlinear term, the unknown input disturbance, the output noise and the sensor fault appear simultaneously. The sensor fault signal will be estimated directly in three such cases.**

Illustrative examples

- **Example 1: Consider the plant in Case 3 with $f(t,x,u)=0$, i.e.**

$$\begin{cases} E\dot{x} = Ax + Bu + B_d d \\ y_1 = C_1 x + d \\ y_2 = C_2 x + D_{d2} d + D_{sf} f_s, \end{cases} \tag{29}$$

where

$$E = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, B_d = \begin{bmatrix} 0.5 & -1 \\ 2.5 & -1 \\ 0 & 0 \end{bmatrix}, C_1 = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}, D_{d2} = \begin{bmatrix} 1 & 2 \end{bmatrix}, D_{sf} = 1, d = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0.6sin(100t) + 0.4sin(40t) \\ 1 + 0.2sin(6t) + 0.3cos(5t) \end{bmatrix},$$

$$u = 3sin(t), \quad f_s = \begin{cases} 0.3(t-2) + 4, & t \geq 2, \\ 0, & t < 2. \end{cases} \tag{30}$$

Note: We now have high-frequency noise d_1, slow-varying disturbance d_2 and unbounded sudden sensor fault f_s.

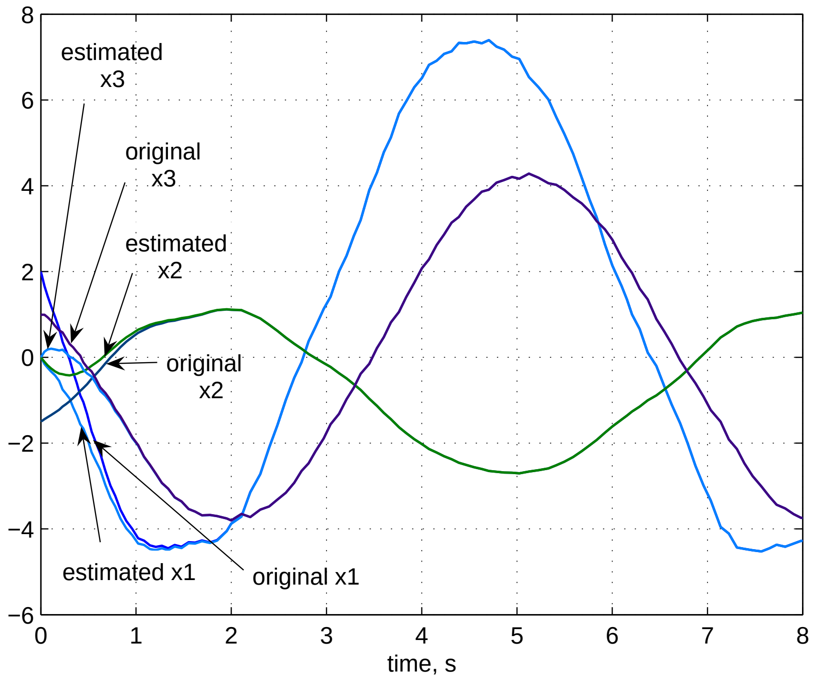


Figure 4: States and their estimates in Example 1

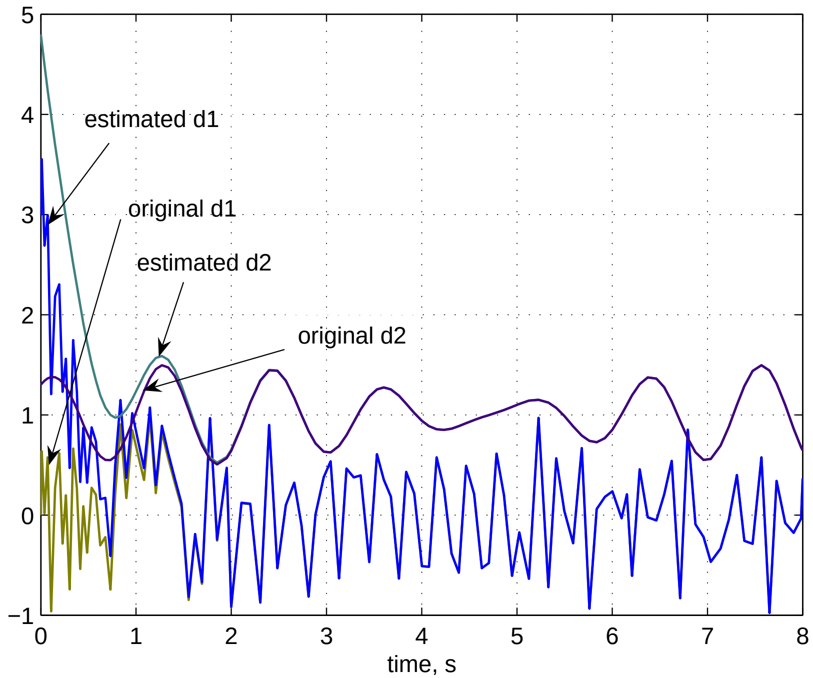


Figure 5: Disturbances and their estimates in Example 1

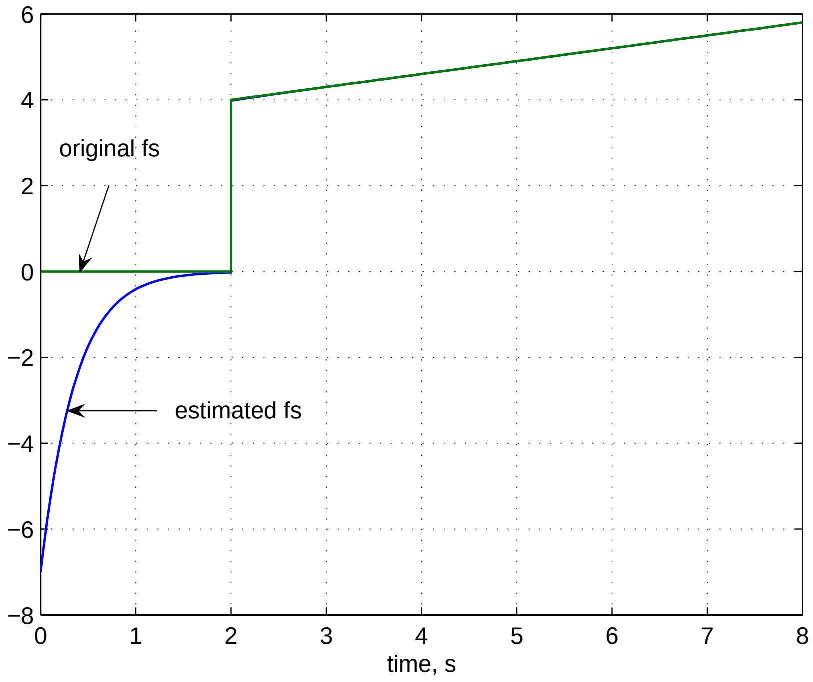


Figure 6: Sensor fault and its estimate in Example 1

- **Example 2 Consider the plant in Case 2, i.e.**

$$\begin{cases} E\dot{x} = Ax + Bu + f(t, x, u) \\ y = Cx + D_d d + D_{sf} f_s, \end{cases} \tag{32}$$

where E, A and B are the same as those in Example 1, and

$$f(t, x, u) = \begin{bmatrix} 0 \\ 0 \\ 0.3358sin(x_3) \end{bmatrix}, \quad u = 3, \quad d = 0.5tx_2x_3,$$

$$f_s = \begin{cases} 4 + 0.8sin[10(t-3)] + 0.3sin[40(t-3)], \ \ t \geq 3, \\ 0, \ \ t < 3, \end{cases}$$

$$C = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}, \ D_d = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \ D_{sf} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}. \tag{33}$$

Note: Now output disturbance d is unbounded and the sensor fault f_s possesses high-frequency component.

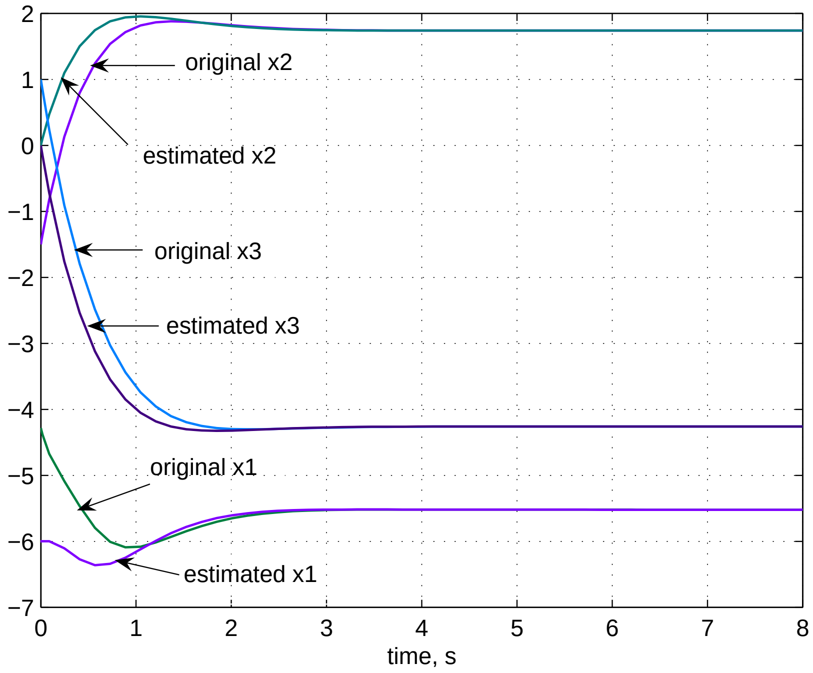


Figure 7: Sates and their estimates in Example 2

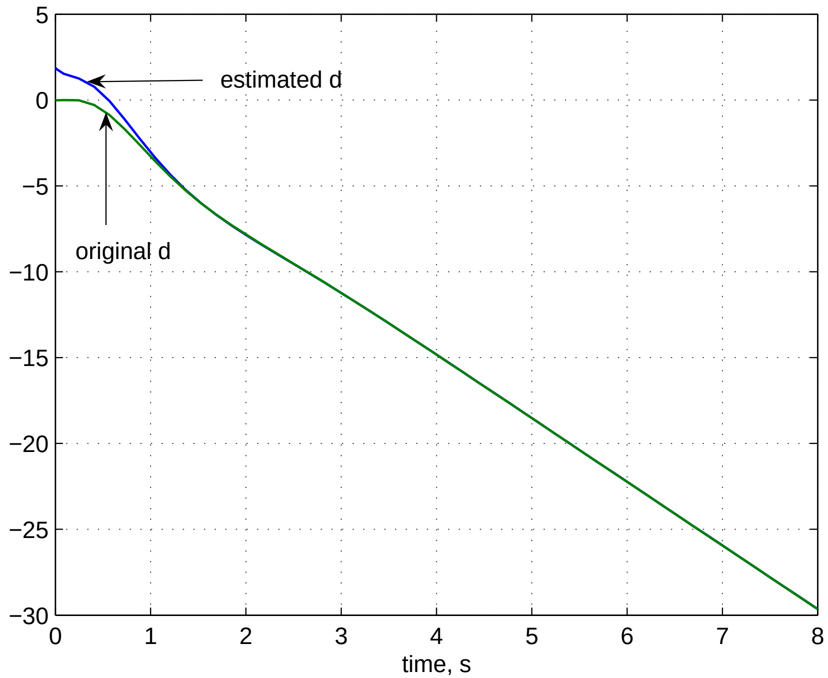


Figure 8: Output disturbance and its estimate in Example 2

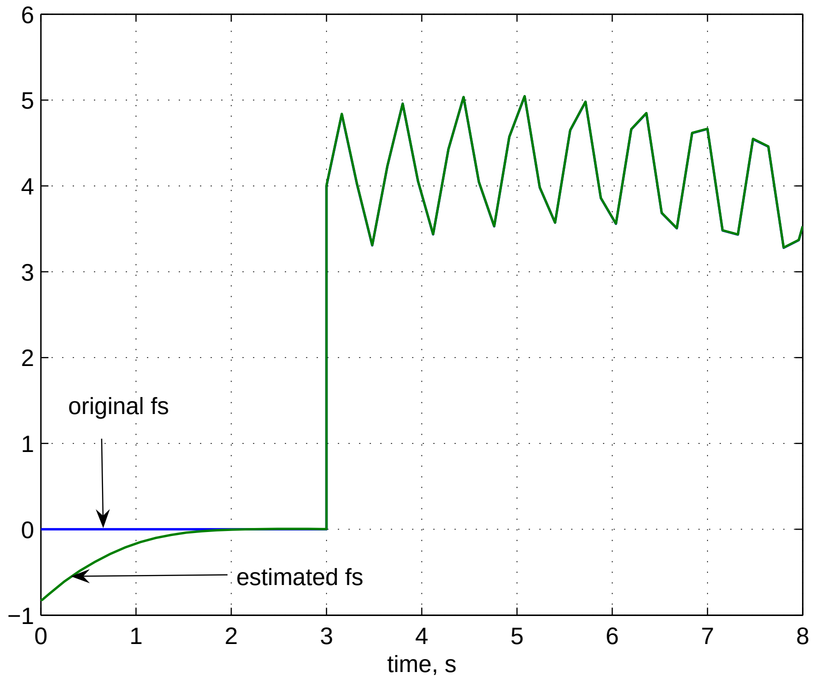
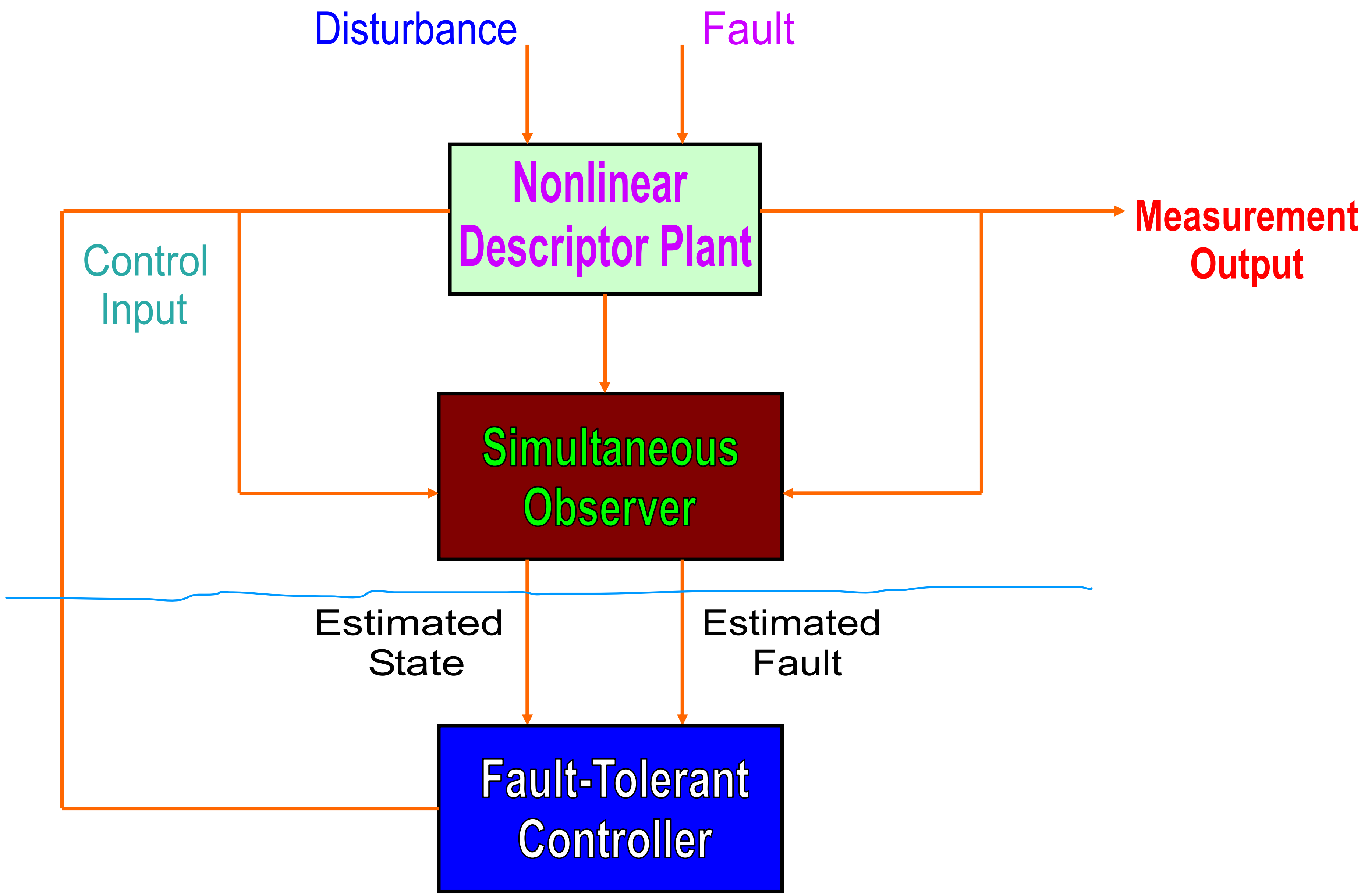


Figure 9: Sensor fault and its estimate in Example 2



Disturbance
Fault
Nonlinear Descriptor Plant
Measurement Output
Control Input
Simultaneous Observer
Estimated State
Estimated Fault
Fault-Tolerant Controller

Some more new stability results on nonlinear singular systems:

- **Generalized Quadratic Stability for Singular systems with nonlinear perturbation, accepted in IEEE AC Feb 2006, to appear 2007.**

- **Continuous stabilization controllers for singular bilinear systems: the state feedback case, Automatica 42, 2006 309-314.**

- **Both papers are joint work with Guoping Lu of Nantong University**

Motivation on Robust Stability

- **The issue of robust quadratic stability for perturbed system is to find a tolerable perturbation bound such that**

  (i) for all admissible parameter perturbations,
  (ii) the system is stable and
  (iii) the associated Lyapunov function is quadratic and deterministic.

- **Robust stability results for both nonsingular and singular systems are available in the literature.**

- **Question:**
  Can we get any robust stability results for continuous-time singular system with time-varying nonlinear perturbations (CSSP) ?

- **In other words, it is known that the generalized quadratic stability implies the global asymptotic stability for linear time-invariant singular systems.**

- **Does generalized quadratic stability imply the global asymptotic stability for this *time-varying* CSSP?**

- **Similar Results published before ??**
  A. Rehm and F. Allgŏwer (2000) published a small paper, where the system matrices are time-invariant and uncertainties are state-independent. Their approaches cannot apply to those system with time-varying and state-dependent uncertainties.

Difficulties

- **One of the difficulties lies in that the existence and uniqueness of the solution for nonlinear singular systems is still an open problem and has not been fully investigated.**

- **In addition, the standard Lyapunov stability theory cannot be applied to CSSP directly and the open problems remains to be important and challenging.**

- **This open problem is addressed here in this paper,**

  (i) sufficient condition for the open problem is presented,

  (ii)Necessary and sufficient condition is obtained in terms of a convex optimization LMI, under which the maximal perturbation bound is obtained to ensure generalized quadratic stability for CSSP.

Linear Matrix Inequality (LMI) optimization

For convenience and compactness, let

$$\mathcal{T}(\gamma, Q, \Gamma_1, \Gamma_2) := \begin{bmatrix} -Q & \Gamma_1' & \Gamma_2' \\ \Gamma_1 & I - Q & 0 \\ \Gamma_2 & 0 & -\gamma I \end{bmatrix}, \tag{39}$$

where γ is a scalar, Q, Γ and Γ_1 are matrices with appropriate dimensions,
I is identity matrix with an appropriate dimension.
Matrix inequality $\mathcal{T}(\gamma, Q, \Gamma_1, \Gamma_2) < 0$ is a unified LMI structure.

The dimension of (39) will be different according to the control designs of state feedback, static output feedback and dynamic output feedback, respectively.

All the following proposed topics can be transformed into an LMI convex optimization problem and can be solved by efficient interior-point algorithms

Consider the following continuous-time singular system with time-varying nonlinear perturbations (CSSP).

$$E\dot{x} = Ax + f(t,x), \quad x(0) = x_0, \tag{40}$$

where $x \in \mathbf{R}^n$ **is the system state,** $A, E \in \mathbf{R}^{n\times n}$ **are constant matrices;** E **may be singular. Without loss of generality, we shall assume that** $0 < \text{rank}(E) = r < n$**;** $x(0) = x_0$ **is a compatible initial condition;** $f = f(t,x) \in \mathbf{R}^n$ **is vector-valued time-varying nonlinear perturbation with** $f(t,0) = 0$ **for all** $t \geq 0$ **and satisfies the following Lipschitz condition for all** $(t,x), (t,\tilde{x}) \in \mathbf{R} \times \mathbf{R}^n$**.**

$$\|f(t,x) - f(t,\tilde{x})\| \leq \alpha \|F(x-\tilde{x})\|, \tag{41}$$

where F **is a constant matrix with appropriate dimension,** α **is a positive scalar. Consequently, from (41), we have**

$$\|f(t,x)\| \leq \alpha \|Fx\|. \tag{42}$$

For convenience, the above f **is called (tolerable) Lipschitz perturbation, or tolerable perturbation in this paper. In the sequel, we always assume that the perturbation** f **satisfies condition (41).**

Robust Stability has not been fully investigated for the above system.

Definition 1 ***The system is regular and impulse free.***

a) the pair (E, A) ***is said to be regular if det***$(sE - A)$ ***is not identical zero.***

b) the pair (E, A) ***is said to be impulse free if deg(det***$(sE-A))$ ***=rank***(E)***.***

Definition 2 ***System (40) is said to enjoy a Lyapunov-like property (with degree*** α***) if there exists a matrix*** P ***such that*** $E^T P = P^T E \geq 0$ ***and***

$$\Delta := [Ax + f(t,x)]^T Px + x^T P^T [Ax + f(t,x)] < 0 \qquad (43)$$

for all tolerable perturbations (42) and $(t, x) \in \mathbf{R} \times (\mathbf{R}^n - \{0\})$***.***

Lemma 3 ***If system (40) enjoys a Lyapunov-like property, then***

i) the nominal system of (40) (that is, $E\dot{x} = Ax$***) is regular and impulse free;***

ii) for any given initial condition $x(0)$ ***and for all tolerable perturbations (42), the solution*** $x = x(t)$ ***of system (40) is globally exponentially stable.***

Since there exists a matrix P **such that (43) holds, then choosing the Lyapunov function candidate as follows:**

$$V(x) = x^T E^T P x, \tag{46}$$

the derivative of V **along system (40) yields**

$$\dot{V}(x(t)) = \Delta < 0, \quad \forall (t, x) \in \mathbf{R} \times (\mathbf{R^n} - \{0\}) \tag{47}$$

under constraint (42). ...

Hence, we can show that the solution of system (40) is globally exponentially stable.

Remark

In the above proof, the standard Lyapunov stability theory cannot be applied directly. The reason is that the standard Lyapunov function is positive definite while the quadratic Lyapunov function candidate here is positive semi-definite (not positive definite) for a singular system, see $E^T P = P^T E \geq 0$

A new definition for CSSP (equivalent to definition 2)

Definition 3 ***System (40) is said to be generalized quadratically stable with degree*** α ***if it enjoys a Lyapunov-like property.***

Theorem 1 ***Singular system (40) with the perturbation f satisfying the constraint (41) is generalized quadratically stable with degree α if and only if there exist a positive scalar ($= \alpha^2$) and a matrix P such that the following convex optimization problem on α^2 and P is solvable.***

$$\begin{aligned} &\text{minimize} && -\alpha^2 \\ &\text{subject to} && E^T P = P^T E \geq 0 \\ & && \begin{pmatrix} A^T P + P^T A + \alpha^2 F^T F & P^T \\ P & -I \end{pmatrix} < 0. \end{aligned} \tag{48}$$

Example 1 ***Consider the following nonlinear singular system.***

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \dot{x}(t) = \begin{pmatrix} -1 & 0 & -3 \\ 3 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix} x(t) + \alpha f(t, x(t)), \tag{49}$$

where $x(t) = \begin{pmatrix} x_1(t) & x_2(t) & x_3(t) \end{pmatrix}^T \in \mathbf{R}^3$***,*** α ***is a positive scalar,***

$$f(t, x(t)) = \begin{pmatrix} sin[x_1(t) + x_2(t)] \\ sin[x_2(t) + x_3(t)] \\ sin[x_3(t) + x_1(t)] \end{pmatrix}.$$

For $f(t, x(t))$**, we have**

$$\begin{aligned} \|f(t, x(t))\|^2 &= \alpha^2 \left\{ sin^2[x_1(t) + x_2(t)] + sin^2[x_2(t) + x_3(t)] + sin^2[x_3(t) + x_1(t)] \right\} \\ &\leq \alpha^2 \left\{ [x_1(t) + x_2(t)]^2 + [x_2(t) + x_3(t)]^2 + [x_3(t) + x_1(t)]^2 \right\} \\ &= \alpha^2 x^T(t) G x(t), \end{aligned} \tag{50}$$

where $G = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$**. The tolerable perturbation bound** α **is given by**

The tolerable perturbation bound α is given by $\alpha_{\max} = 0.4254$. It is worth pointing out that the robust stability issue of this example cannot be implemented by the technique of other work, since the corresponding perturbations have to be bounded by modulus matrix with time-invariant structure.

***Example 3** Consider the numerical examples in (Chen and Han 1994) and (Yan and Lam 2001). That is, (i) the system (40) with* $E = F = I_2$ *and* $A = \begin{pmatrix} -3 & -2 \\ 1 & 0 \end{pmatrix}$*;* *(ii) the system (40) with* $E = F = I_5$ *and*

$$A = \begin{pmatrix} -0.2010 & 0.7550 & 0.3510 & -0.0750 & 0.0330 \\ -0.1490 & -0.6960 & -0.1600 & 0.1100 & -0.0480 \\ 0.0810 & 0.0040 & -0.1890 & -0.0030 & 0.0010 \\ -0.1730 & 0.8020 & 0.2510 & -0.8040 & 0.0560 \\ 0.0920 & -0.4670 & -0.1270 & 0.0750 & -1.1620 \end{pmatrix}. \qquad (51)$$

For case (i), $\alpha_{\max} = 0.5399 > 0.4495$ *in (Chen and Han 1994) and* 0.535 *in (Kim 1995) for the same example.*

For case (ii), the tolerable bound is $\alpha_{\max} = 0.1116 > 0.0929$ *in (Chen and Han 1994)*

For case (ii), we have that $\sum_{i=1}^{5} \alpha_{i\max} = 0.3432$ *while* $\sum_{i=1}^{5} \alpha_{i\max} = 0.1490$ *in (Yan and Lam 2001)*

Computation of the (J, J')-Lossless Factorization for General Rational Matrices

- **Joint work with Dr. Delin Chu of Department of Mathematics, National University of Singapore.**
- **appeared in SIAM Optimization & Control, 2005. Sept**

- **Motivations**
- **Problem Formulation**
- **New Result**
- **Conclusions**

Motivations

Two important approach for the H_∞ control:

- **The LMI approach: it employs methods of semidefinite programming to compute the desired optimal H_∞-controllers.**

  This is very attractive, because easy-to-use methods for semidefinite programming are available.

  However, the computational complexity of this approach for a control plant with dimension n is up to $O(n^6)$, which is rather high.

- **The (J, J')-lossless factorization approach:**

  it provides a simple and unified framework of H_∞ control problem from classical network theory point of view.

  This approach leads that the optimal controllers for H_∞ control problem can be obtained by solving the asociated Riccati equations, for which numerically reliable methods have been developed.

  However, these related Riccati equations may become very ill-conditioned when the computed optimal H_∞-norm approaches to the exact optimal H_∞-norm, which leads that these Riccati equations are very difficult to solve.

Therefore, although every existing approach, including the LMI approach and the (J, J')-lossless factorization approach, has its own approach of solving the H_∞ control problem, many numerical problems associated with it remain to be studied.

Notation:

- $J \in \mathbf{R}^{p\times p}, J' \in \mathbf{R}^{m\times m}$: **two given symmetric matrices;**
- $M \geq 0 : M$ **is symmetric and positive semi-definite;**
- $\rho(M, N)$**: the spectral radius of the pencil** $-sM + N$**, and** $\rho(N) := \rho(I, N)$**;**
- $\mathbf{C}_0, \mathbf{C}_+$**: imaginary axis and open right half complex plane, respectively;**
- $\mathcal{R}^{p\times m}(s), \mathcal{RL}_\infty^{p\times m}(s)$**: set of** $p \times m$ **real rational matrices and set of** $p \times m$ **proper real rational matrices having no poles on** $\mathbf{C}_0$**, respectively;**
- $G(s) = \left[\begin{array}{c|c} -sE + A & B \\ \hline C & D \end{array}\right]$ **means that** $G(s)$ **has a realization** $G(s) = D + C(sE - A)^{-1}B$**.**

Problem Formulation

Definition 1 ***(i) A matrix*** $\Theta(s) \in \mathcal{RL}_\infty^{p\times m}(s)$ ***is*** (J, J')***-unitary if***

$$\Theta^T(-s)J\Theta(s) = J', \ \forall s \in \mathbf{C}.$$

(ii) A matrix $\Theta(s) \in \mathcal{RL}_\infty^{p\times m}(s)$ ***is*** (J, J')***-lossless if it is*** (J, J')***-unitary and***

$$\Theta^T(\bar{s})J\Theta(s) \leq J', \ \forall s \in \mathbf{C}_0 \cup \mathbf{C}_+,$$

where $\bar{s}$ ***is the complex conjugate of*** s***.***

Definition 2 $G(s) \in \mathcal{R}^{p\times m}(s)$ ***has a*** (J, J')***-lossless factorization if it can be represented as a product***

$$G(s) = \Theta(s)\Xi(s),$$

where $\Theta(s) \in \mathcal{RL}_\infty^{p\times m}(s)$ ***is*** (J, J')***-lossless, and*** $\Xi(s) \in \mathcal{R}^{m\times m}(s)$ ***has neither zeros nor poles in*** $\mathbf{C}_+$***.***

The main existing result for the (J, J')-lossless factorization of general proper rational matrices can be summarized as follows:

Assume that $G(s) = \left[\begin{array}{c|c} -sI + A & B \\ \hline C & D \end{array}\right] \in \mathcal{RL}_{\infty}^{p\times m}(s)$ is left invertible. Here $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$, $C \in \mathbf{R}^{p\times n}$ and $D \in \mathbf{R}^{p\times m}$. Then there exist an orthogonal matrix S and a nonsingular matrix T such that

$$S \begin{bmatrix} -sI + A & B \\ C & D \end{bmatrix} T = \begin{array}{c} \begin{array}{ccc} n - n_{0\infty} & n_{0\infty} & m \end{array} \\ \begin{bmatrix} -sE_{nf} + A_{nf} & 0 & 0 \\ \star & -sE_{11} + A_{11} & A_{12} \\ \star & A_{21} & A_{22} \end{bmatrix} \begin{array}{l} \\ \}n_{0\infty} \\ \}m \end{array} \end{array},$$

where E_{nf} is of full column rank, E_{11} is nonsingular,

$$\begin{aligned} &\mathrm{rank}(-sE_{nf} + A_{nf}) = n - n_{0\infty}, \quad \forall s \in \mathbf{C}_0, \\ &\mathrm{rank} \begin{bmatrix} -sE_{11} + A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = n_{0\infty} + m, \ \ \forall s \in \mathbf{C} \backslash \mathbf{C}_0. \end{aligned}$$

Partition S and T into

$$S = \begin{array}{c} \begin{matrix} n & p \end{matrix} \\ \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \\ S_{31} & S_{32} \end{bmatrix} \begin{matrix} \\ \}n_{0\infty} \\ \}m \end{matrix} \end{array}, \quad T = \begin{array}{c} \begin{matrix} n-n_{0\infty} & n_{0\infty} & m \end{matrix} \\ \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \end{bmatrix} \begin{matrix} \}n \\ \}m \end{matrix} \end{array}.$$

Further, let the columns of full column rank matrix $\begin{array}{c} r \\ \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} \begin{matrix} \}n \\ \}n \\ \}m \end{matrix} \end{array}$ **span the stable eigenspace of the matrix pencil**

$$\begin{bmatrix} -sI + A & 0 & B \\ -C^T JC & -sI - A^T & -C^T JD \\ D^T JC & B^T & D^T JD \end{bmatrix},$$

and there exists a stable matrix $\Lambda \in \mathbf{R}^{r \times r}$ such that

$$\begin{bmatrix} A & 0 & B \\ -C^T JC & -A^T & -C^T JD \\ D^T JC & B^T & D^T JD \end{bmatrix} \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} = \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} \Lambda.$$

Theorem 1 ***(c.f. [1]) Let*** $G(s) =: \left[\begin{array}{c|c} -sI+A & B \\ \hline C & D \end{array}\right] \in \mathcal{RL}_\infty^{p\times m}(s)$ ***be stabilizable and detectable. Then*** $G(s)$ ***has a*** (J,J')***-lossless factorization if and only if the following conditions hold.***

(i) $G(s)$ ***is left invertible;***

(ii) There exists a $D_0 \in \mathbf{R}^{m\times m}$ ***such that*** $D_0^T S_{32} J S_{32}^T D_0 = J'$***;***

(iii) $r + n_{0\infty} = n$***,*** $\left[\begin{array}{cc} L_1 & T_{12} \end{array}\right]$ ***is nonsingular,***

$$X := \left[\begin{array}{cc} L_2 & 0 \end{array}\right] \left[\begin{array}{cc} L_1 & T_{12} \end{array}\right]^{-1} \geq 0,$$

and the algebraic Riccati equation

$$YA^T + AY + YC^TJCY = 0$$

has a solution $Y \geq 0$ ***such that*** $A + YC^TJC$ ***is stable;***

(vi) $\rho(XY) < 1$***.***

Moreover, if the above conditions hold, then a (J, J')***-lossless factorization is given by the factors*** $\Theta(s)$ ***and*** $\Xi(s)$***:***

$$\begin{aligned}
\Theta(s) &= \left[\begin{array}{cc|c} -sI+\Lambda & 0 & Z_1 \\ 0 & -sI+A & Z_2 \\ \hline CL_1+DL_3 & C & -S_{32}^T \end{array}\right] D_0, \\
\Xi(s) &= -(J')^{-1}D_0^T \\
&\quad \times \left[\begin{array}{c|c} -sI+A+YC^TJC & B+YC^TJD \\ \hline (S_{31}X+S_{32}JC)(I-YX)^{-1} & S_{32}JD \end{array}\right],
\end{aligned}$$

where

$$\begin{aligned}
Z_1 &= -\left[\, I_r \ \ 0 \,\right]\left[\, L_1 - YL_2 \ \ T_{12} \,\right]^{-1}(S_{31}^T + YC^TJS_{32}^T), \\
Z_2 &= (I-YX)^{-1}Y(XS_{31}^T + C^TJS_{32}^T).
\end{aligned}$$

Obviously, Theorem 1 excludes all non-proper rational matrices.

In [2], the (J, J')-lossless factorization problem for general rational matrices has been considered using the descriptor-form representation approach. The results in [2] are based on:

(i) a realization of $G(s) = \left[\begin{array}{c|c} -sE + A & B \\ \hline C & D \end{array}\right]$**, which is in standard form, i.e., there do not exist nonsingular matrices** M **and** N **and an integer** $r > 0$ **such that** $M(-sE + A)N = \begin{bmatrix} -s\hat{E} + \hat{A} & 0 \\ 0 & I_r \end{bmatrix}$**;**

(ii) $E^2 = E$**;**

(iii) the generalized Lyapunov equation

$$\begin{cases} AYE^T + EYA^T + EYC^TJCYE^T = 0, \quad E \text{ is singular}, \\ A^T + C^TJCYE^T - sE^T \text{ is nonsingular for all } s \in \mathbf{C}_+, \\ EYE^T \geq 0, \\ \text{the null space of } YE^T \text{ contain the eigenspace of} \\ \qquad -sE^T + A^T \text{ corresponging to the eigenvalues} \\ \qquad\qquad\qquad \text{on } \mathbf{C}_0 \cup \{\infty\}, \end{cases} \tag{1}$$

(vi) existence of matrices $D_\pi \in \mathbf{R}^{m\times m}$**,** $K \in \mathbf{R}^{m\times n}$ **and a** (J, J')**-lossless matrix** D_c **satisfying**

$$\left[\begin{array}{cc} \tilde{C} & D \end{array}\right] = D_c \left[\begin{array}{cc} K & D_\pi \end{array}\right], \ \tilde{C} \text{ is known.} \tag{2}$$

However,

- **The computation of a realization of $G(s) \in \mathcal{R}^{p\times m}(s)$, which is in the standard form and satisfies $E^2 = E$, is very ill-conditioned and cannot be obtained in a numerically reliable manner. This issue is easy to understand, for example, let us consider a very simple example. Let $G(s)$ be of the form**

$$G(s) = \left[\begin{array}{ccc|c} -sE_{11} + A_{11} & A_{12} & A_{13} & B_1 \\ A_{21} & 0 & 0 & B_2 \\ 0 & 0 & A_{33} & B_3 \\ \hline C_1 & C_2 & C_3 & D \end{array}\right],$$

where E_{11} and A_{33} are nonsingular.

Then, in order to get a realization of $G(s)$ which is in the standard form, we have to compute A_{33}^{-1}. But, the computation of A_{33}^{-1} is numerically unstable and will contain large error if A_{33} is ill-conditioned.

- **The generalized Lyapunov equation (1) is very difficult to solve, and**

  it is not clear under what conditions there exist D_π, K and a (J, J')-lossless matrix D_c satisfying (2) because of the requirement that D_c is (J, J')-lossless.

  Thus, the computation of matrices $D_\pi \in \mathbf{R}^{m\times m}$, $K \in \mathbf{R}^{m\times n}$ and a (J, J')-lossless matrix D_c has to be studied further.

Hence, there is still a lack of numerically reliable methods for solving the (J, J')-lossless factorization problem with general rational matrices.

New Result

Theorem 2 ***Given a rational matrix*** $G(s) \in \mathcal{R}^{p\times m}(s)$***. Let***

$$G(s) = \left[\begin{array}{c|c} -sE + A & B \\ \hline C & 0 \end{array}\right], \quad E, A \in \mathbf{R}^{n\times n}, \quad B \in \mathbf{R}^{n\times m}, \quad C \in \mathbf{R}^{p\times n} \qquad (3)$$

be its minimal realization. There exist non-negative integers n_1***,*** n_2 ***and*** n_3 ***with*** $n_1 + n_2 + n_3 = n$***, orthogonal matrices*** $P, Q, U \in \mathbf{R}^{n\times n}$***,*** $W \in \mathbf{R}^{m\times m}$ ***and*** $V \in \mathbf{R}^{(n_1+n_3)\times(n_1+n_3)}$ ***with*** U ***and*** V ***being partitioned as***

$$U = \begin{array}{c} \begin{array}{cc} n_1+n_2 & n_3 \end{array} \\ \left[\begin{array}{cc} U_{11} & U_{12} \\ U_{21} & U_{22} \end{array}\right] \begin{array}{l} \}n_1+n_2 \\ \}n_3 \end{array} \end{array}, \qquad \mathrm{rank}(U_{11}) = n_1 + n_2,$$

$$V = \begin{array}{c} \begin{array}{cc} n_3 & n_1 \end{array} \\ \left[\begin{array}{cc} V_{11} & V_{12} \\ V_{21} & V_{22} \end{array}\right] \begin{array}{l} \}n_3 \\ \}n_1 \end{array} \end{array}, \qquad \mathrm{rank}(V_{11}) = n_3$$

such that

$$
\begin{bmatrix} U_{11} & 0 & U_{12} \\ 0 & I & 0 \\ U_{21} & 0 & U_{22} \end{bmatrix}
\begin{bmatrix} P & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & P \end{bmatrix}
\begin{bmatrix} -sE+A & B & -sE+A \\ C & 0 & C \\ -sE+A & B & 0 \end{bmatrix}
$$

$$
\times \begin{bmatrix} Q & 0 & 0 \\ 0 & W & 0 \\ 0 & 0 & Q \end{bmatrix}
\begin{bmatrix} I_{n_1+n_2} & 0 & 0 & 0 & 0 \\ 0 & V_{11} & 0 & V_{12} & 0 \\ 0 & 0 & I_m & 0 & 0 \\ 0 & V_{21} & 0 & V_{22} & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}
$$

$$
= \begin{array}{c} \\ \left[\begin{array}{cccccc} n_1 & n_2 & n_3 & n_3 & m-n_3 & n \\ -sE_{11}+A_{11} & -sE_{12}+A_{12} & A_{13} & 0 & B_{12} & \star \\ 0 & -sE_{22}+A_{22} & A_{23} & 0 & B_{22} & \star \\ 0 & A_{32} & A_{33} & B_{31} & 0 & \star \\ C_1 & C_2 & C_3 & 0 & 0 & \star \\ \star & \star & \star & \star & \star & \star \end{array} \right] \end{array}
\begin{array}{l} \\ \}n_1 \\ \}n_2 \\ \}n_3 \\ \}p \\ \}n \end{array}, \qquad (4)
$$

where $\star$ denotes the sub-block which we are not interested in,

and

$$\operatorname{rank}(E_{11}) = n_1, \ \ \operatorname{rank}(E_{22}) = n_2, \ \ \operatorname{rank}(B_{31}) = n_3, \tag{5}$$

$$\operatorname{rank}\begin{bmatrix} -sE_{22}+A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} = n_2 + n_3, \quad \forall s \in \mathbf{C}. \tag{6}$$

The computation of the factorization (4) needs only $O(n^3 + m^3)$ flops.

Assume that $G(s) \in \mathcal{R}^{p\times m}(s)$ with a minimal realization (3) is left invertible. Then

$$\max_{s\in\mathbf{C}} \operatorname{rank}\begin{bmatrix} -sE_{11}+A_{11} & -sE_{12}+A_{12} & A_{13} & B_{12} \\ 0 & -sE_{22}+A_{22} & A_{23} & B_{22} \\ C_1 & C_2 & C_3 & 0 \end{bmatrix}$$

$$= (n+m) - n_3 = n_1 + n_2 + n_3 + (m - n_3),$$

Therefore, the generalized lower triangular form of the pencil is of the form

$$\mathcal{S}\begin{bmatrix} -sE_{11}+A_{11} & -sE_{12}+A_{12} & A_{13} & B_{12} \\ 0 & -sE_{22}+A_{22} & A_{23} & B_{22} \\ C_1 & C_2 & C_3 & 0 \end{bmatrix}\mathcal{T} =$$

$$\begin{array}{c} \begin{array}{cccc} n_1+n_2-n_{0\infty} & n_{0\infty} & n_3 & m-n_3 \end{array} \\ \left[\begin{array}{cccc} -s\mathcal{E}_{nf}+\mathcal{A}_{nf} & 0 & 0 & 0 \\ -s\mathcal{E}_{10}+\mathcal{A}_{10} & -s\mathcal{E}_{11}+\mathcal{A}_{11} & \mathcal{A}_{12} & \mathcal{A}_{13} \\ -s\mathcal{E}_{20}+\mathcal{A}_{20} & \mathcal{A}_{21} & \mathcal{A}_{22} & \mathcal{A}_{23} \\ -s\mathcal{E}_{30}+\mathcal{A}_{30} & \mathcal{A}_{31} & \mathcal{A}_{32} & \mathcal{A}_{33} \end{array}\right] \begin{array}{l} \\ \}n_{0\infty} \\ \}n_3 \\ \}m-n_3 \end{array} \end{array}, \qquad (7)$$

where $\mathcal{S}$ **and** $\mathcal{T}$ **are orthogonal,** $\mathcal{E}_{nf}$ **is of full column rank,** $\mathcal{E}_{11}$ **is nonsingular, and**

$$\operatorname{rank}(-s\mathcal{E}_{nf}+\mathcal{A}_{nf}) = n_1+n_2-n_{0\infty}, \quad \forall s \in \mathbf{C}_0,$$

$$\operatorname{rank}\begin{bmatrix} -s\mathcal{E}_{11}+\mathcal{A}_{11} & \mathcal{A}_{12} & \mathcal{A}_{13} \\ \mathcal{A}_{21} & \mathcal{A}_{22} & \mathcal{A}_{23} \\ \mathcal{A}_{31} & \mathcal{A}_{32} & \mathcal{A}_{33} \end{bmatrix} = n_{0\infty}+m, \quad \forall s \in \mathbf{C}\backslash\mathbf{C}_0.$$

Partitioning $\mathcal{S}$ and $\mathcal{T}$ into

$$\begin{cases} \mathcal{S} = \begin{bmatrix} \overset{n_1}{\mathcal{S}_{11}} & \overset{n_2}{\mathcal{S}_{12}} & \overset{p}{\mathcal{S}_{13}} \\ \mathcal{S}_{21} & \mathcal{S}_{22} & \mathcal{S}_{23} \\ \mathcal{S}_{31} & \mathcal{S}_{32} & \mathcal{S}_{33} \\ \mathcal{S}_{41} & \mathcal{S}_{42} & \mathcal{S}_{43} \end{bmatrix} \begin{matrix} \} n_{0\infty} \\ \} n_3 \\ \} m - n_3 \end{matrix} , \\ \\ \mathcal{T} = \begin{bmatrix} \overset{n_1 + n_2 - n_{0\infty}}{\mathcal{T}_{11}} & \overset{n_{0\infty}}{\mathcal{T}_{12}} & \overset{n_3}{\mathcal{T}_{13}} & \overset{m - n_3}{\mathcal{T}_{14}} \\ \mathcal{T}_{21} & \mathcal{T}_{22} & \mathcal{T}_{23} & \mathcal{T}_{24} \\ \mathcal{T}_{31} & \mathcal{T}_{32} & \mathcal{T}_{33} & \mathcal{T}_{34} \\ \mathcal{T}_{41} & \mathcal{T}_{42} & \mathcal{T}_{43} & \mathcal{T}_{44} \end{bmatrix} \begin{matrix} \} n_1 \\ \} n_2 \\ \} n_3 \\ \} m - n_3 \end{matrix} . \end{cases} \tag{8}$$

Obviously, the factorization (7) has isolated the zeros of $G(s)$ on $\mathbf{C}_0$ and at infinity to $\begin{bmatrix} -s\mathcal{E}_{11} + \mathcal{A}_{11} & \mathcal{A}_{12} & \mathcal{A}_{13} \\ \mathcal{A}_{21} & \mathcal{A}_{22} & \mathcal{A}_{23} \\ \mathcal{A}_{31} & \mathcal{A}_{32} & \mathcal{A}_{33} \end{bmatrix}$.

Let the columns of full column rank matrix

$$\begin{bmatrix} L_1^T & L_2^T & L_3^T & L_4^T & L_5^T & L_6^T \end{bmatrix}^T$$

with $L_1, L_3 \in \mathbf{R}^{n_1\times r}$**,** $L_2, L_4 \in \mathbf{R}^{n_2\times r}$**,** $L_5 \in \mathbf{R}^{n_3\times r}$ **and** $L_6 \in \mathbf{R}^{(m-n_3)\times r}$ **span the stable eigenspace of the pencil**

$$\begin{bmatrix} -sE_{11}+A_{11} & -sE_{12}+A_{12} & 0 & 0 & A_{13} & B_{12} \\ 0 & -sE_{22}+A_{22} & 0 & 0 & A_{23} & B_{22} \\ -C_1^TJC_1 & -C_1^TJC_2 & -(sE_{11}+A_{11})^T & 0 & -C_1^TJC_3 & 0 \\ -C_2^TJC_1 & -C_2^TJC_2 & -(sE_{12}+A_{12})^T & -(sE_{22}+A_{22})^T & -C_2^TJC_3 & 0 \\ C_3^TJC_1 & C_3^TJC_2 & A_{13}^T & A_{23}^T & C_3^TJC_3 & 0 \\ 0 & 0 & B_{12}^T & B_{22}^T & 0 & 0 \end{bmatrix},$$

which gives

$$\begin{bmatrix} A_{11} & A_{12} & 0 & 0 & A_{13} & B_{12} \\ 0 & A_{22} & 0 & 0 & A_{23} & B_{22} \\ -C_1^TJC_1 & -C_1^TJC_2 & -A_{11}^T & 0 & -C_1^TJC_3 & 0 \\ -C_2^TJC_1 & -C_2^TJC_2 & -A_{12}^T & -A_{22}^T & -C_2^TJC_3 & 0 \\ C_3^TJC_1 & C_3^TJC_2 & A_{13}^T & A_{23}^T & C_3^TJC_3 & 0 \\ 0 & 0 & B_{12}^T & B_{22}^T & 0 & 0 \end{bmatrix} \begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \\ L_5 \\ L_6 \end{bmatrix}$$

$$= \begin{bmatrix} E_{11} & E_{12} & 0 & 0 & 0 & 0 \\ 0 & E_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & E_{11}^T & 0 & 0 & 0 \\ 0 & 0 & E_{12}^T & E_{22}^T & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \\ L_5 \\ L_6 \end{bmatrix} \Delta, \qquad (9)$$

where $\Delta \in \mathbf{R}^{r\times r}$ **is stable.**

Theorem 3 ***Given*** $G(s) \in \mathcal{R}^{p\times m}(s)$ ***with a minimal realization (3). Assume that the factorizations in Theorem 2. have been determined. Then*** $G(s)$ ***has a*** (J, J')***-lossless factorization if and only if the following conditions hold:***

(a) $G(s)$ ***is left invertible;***

(b) There exists a nonsingular matrix $\mathcal{D}_0 \in \mathbf{R}^{m\times m}$ ***such that***

$$\mathcal{D}_0^T \begin{bmatrix} \mathcal{S}_{33} \\ \mathcal{S}_{43} \end{bmatrix} J \begin{bmatrix} \mathcal{S}_{33}^T & \mathcal{S}_{43}^T \end{bmatrix} \mathcal{D}_0 = J'. \tag{10}$$

(c)

$$r + n_{0\infty} = n_1 + n_2, \quad \begin{bmatrix} L_1 & \mathcal{T}_{12} \\ L_2 & \mathcal{T}_{22} \end{bmatrix} \text{ is nonsingular,} \tag{11}$$

$$\begin{bmatrix} E_{11}L_1 + E_{12}L_2 & E_{11}\mathcal{T}_{12} + E_{12}\mathcal{T}_{22} \\ E_{22}L_2 & E_{22}\mathcal{T}_{22} \end{bmatrix}^T \begin{bmatrix} L_3 & 0 \\ L_4 & 0 \end{bmatrix} \geq 0. \tag{12}$$

and the algebraic Riccati equation

$$E_{11}\mathcal{Y}_{11}A_{11}^T + A_{11}\mathcal{Y}_{11}E_{11}^T + E_{11}\mathcal{Y}_{11}C_1^T J C_1 \mathcal{Y}_{11}E_{11}^T = 0 \tag{13}$$

has a solution $\mathcal{Y}_{11} \geq 0$ ***such that the pencil*** $-sE_{11} + A_{11} + E_{11}\mathcal{Y}_{11}C_1^T J C_1$ ***is stable.***

(d)

$$\rho(\begin{bmatrix} L_1 & \mathcal{T}_{12} \\ L_2 & \mathcal{T}_{22} \end{bmatrix}, \begin{bmatrix} \mathcal{Y}_{11}E_{11}^T L_3 & 0 \\ 0 & 0 \end{bmatrix}) < 1. \tag{14}$$

Furthermore, in the case that the conditions (a), (b), (c) and (d) above hold, if we define the following two QR factorizations

$$\hat{\mathcal{W}} \begin{bmatrix} S_{21}^T \\ S_{22}^T \end{bmatrix} = \begin{bmatrix} 0 \\ \mathcal{R}_{\hat{\mathcal{W}}} \end{bmatrix}, \quad \hat{\mathcal{W}}\hat{\mathcal{W}}^T = I, \quad \operatorname{rank}(\mathcal{R}_{\hat{\mathcal{W}}}) = n_{0\infty}, \tag{15}$$

$$\begin{cases} \tilde{\mathcal{W}}\left(\begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & I_r & 0 \end{bmatrix} \begin{bmatrix} E_{11}\mathcal{Y}_{11}E_{11}^T L_3 \\ E_{11}L_1 + E_{12}L_2 \\ E_{22}L_2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ \mathcal{R}_{\tilde{\mathcal{W}}} \end{bmatrix}, \\ \tilde{\mathcal{W}}\tilde{\mathcal{W}}^T = I, \quad \operatorname{rank}(\mathcal{R}_{\tilde{\mathcal{W}}}) = r, \end{cases} \tag{16}$$

and partition

$$\begin{bmatrix} \tilde{\mathcal{W}} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \hat{\mathcal{W}} \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ I & I & 0 \\ 0 & 0 & I_{n_2} \end{bmatrix} =: \begin{array}{c} \begin{array}{cc} n_1 & n_1+n_2 \end{array} \\ \begin{bmatrix} \mathcal{W}_{11} & \mathcal{W}_{12} \\ \mathcal{W}_{21} & \mathcal{W}_{22} \end{bmatrix} \end{array} \begin{array}{l} \}n_1 \\ \}n_1+n_2 \end{array}, \tag{17}$$

then a (J,J')***-lossless factorization of*** $G(s)$ ***is given by the factors*** $\Theta(s)$ ***and*** $\Xi(s)$***:***

$$\Theta(s) = \left[\begin{array}{cc|c} -sE_\Theta + A_\Theta & 0 & \mathcal{Z}_1 \\ 0 & \mathcal{W}_{11}(-sE_{11} + A_{11}) & \mathcal{Z}_2 \\ \hline C_1L_1 + C_2L_2 + C_3L_5 & C_1 & -\begin{bmatrix} \mathcal{S}_{33} \\ \mathcal{S}_{43} \end{bmatrix}^T \end{array}\right] \mathcal{D}_0, \qquad (18)$$

$$\Xi(s) = -(J')^{-1}\mathcal{D}_0^T \left[\begin{array}{c|c} sE_\Xi + A_\Xi & B_\Xi \\ \hline C_\Xi & 0 \end{array}\right] W^T, \qquad (19)$$

where

$$-sE_\Theta + A_\Theta = \begin{bmatrix} I_r & 0 \end{bmatrix} \hat{\mathcal{W}} \begin{bmatrix} E_{11} & E_{12} \\ 0 & E_{22} \end{bmatrix} \begin{bmatrix} L_1 - \mathcal{Y}_{11} E_{11}^T L_3 \\ L_2 \end{bmatrix} (-sI + \Delta),$$

$$\mathcal{Z}_1 = -\begin{bmatrix} I_r & 0 \end{bmatrix} \hat{\mathcal{W}} \left(\begin{bmatrix} \mathcal{S}_{31} & \mathcal{S}_{32} \\ \mathcal{S}_{41} & \mathcal{S}_{42} \end{bmatrix}^T + \begin{bmatrix} E_{11}\mathcal{Y}_{11}C_1^T \\ 0 \end{bmatrix} J \begin{bmatrix} \mathcal{S}_{33} \\ \mathcal{S}_{43} \end{bmatrix}^T \right),$$

$$\mathcal{Z}_2 = -\mathcal{W}_{12} \begin{bmatrix} \mathcal{S}_{31} & \mathcal{S}_{32} \\ \mathcal{S}_{41} & \mathcal{S}_{42} \end{bmatrix}^T + (\mathcal{W}_{11} - \mathcal{W}_{12} \begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix}) E_{11}\mathcal{Y}_{11}C_1^T J \begin{bmatrix} \mathcal{S}_{33} \\ \mathcal{S}_{43} \end{bmatrix}^T,$$

$$-sE_\Xi + A_\Xi = \left(\begin{bmatrix} -sE_{11} + A_{11} & -sE_{12} + A_{12} & A_{13} \\ 0 & -sE_{22} + A_{22} & A_{23} \\ 0 & A_{32} & A_{33} \end{bmatrix} + \begin{bmatrix} E_{11}\mathcal{Y}_{11}C_1^T J \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} C_1 & C_2 & C_3 \end{bmatrix} \right) \times \begin{bmatrix} L_1 - \mathcal{Y}_{11}E_{11}^T L_3 & \mathcal{T}_{12} & 0 \\ L_2 & \mathcal{T}_{22} & 0 \\ 0 & 0 & I \end{bmatrix},$$

$$B_\Xi = \begin{bmatrix} 0 & B_{12} \\ 0 & B_{21} \\ B_{31} & 0 \end{bmatrix},$$

$$C_\Xi = \begin{bmatrix} \mathcal{S}_{31} & \mathcal{S}_{32} & \mathcal{S}_{33}J \\ \mathcal{S}_{41} & \mathcal{S}_{42} & \mathcal{S}_{43}J \end{bmatrix} \begin{bmatrix} L_3 & 0 & 0 \\ L_4 & 0 & 0 \\ C_1L_1 + C_2L_2 & C_1\mathcal{T}_{12} + C_2\mathcal{T}_{22} & C_3 \end{bmatrix}.$$

Algorithm 1

Input: $G(s) \in \mathcal{R}^{p\times m}(s)$ ***with a minimal realization (3).***

Output: ***A*** (J, J')***-lossless factorization*** $G(s) = \Theta(s)\Xi(s)$ ***of*** $G(s)$***, if possible.***

Step 1. Compute the $\max_{s\in\mathbf{C}} \operatorname{rank} \begin{bmatrix} -sE + A & B \\ C & 0 \end{bmatrix}$ ***using the generalized lower triangular form [3] of the pencil***

$$\begin{bmatrix} -sE + A & B \\ C & 0 \end{bmatrix},$$

if it equals to $n + m$***, continue the process. Otherewise, stop;***

Step 2. Compute the factorizations (4), (7) and (9);

Step 3. Solve the algebraic Riccati equation (13);

Step 4. Verify the conditions (10), (11), (12) and (14). If these conditions hold, continue. Otherwise, stop;

Step 5. Compute QR factorizations (15) and (16) and then do the partitioning (17).

Step 6. Compute the factors $\Theta(s)$ ***and*** $\Xi(s)$ ***by (18) and (19). Output*** $\Theta(s)$ ***and*** $\Xi(s)$ ***and then stop.***

We comment on Algorithm 1 as follows:

- **The basis of Algorithm 1 is the factorization (4) whose computation is numerically backward stable;**
- **Steps 1, 2, 4 and 5 are all implemented by only orthogonal transformations which are numerically backward stable;**
- **The algebraic Riccati equation (13) in Step 3 can be solved by MATLAB code** *care.m* **which is known to be numerically reliable;**
- J' **is symmetric, its inverse in Step 6 can be computed by SVDs or QR factorizations which are numerically reliable. Moreover, its computation has no effect on Steps 1–5. Here we wish to emphasize that it is almost impossible to avoid the computation of** $(J')^{-1}$ **in** (J, J')**-lossless factorization problem.**

Therefore, Algorithm 1 can be implemented in a numerically reliable manner.

Conclusions

We have presented necessary and sufficient solvability conditions and developed a numerical algorithm based on a generalized eigenvalue approach for the (J, J')-lossless factorization of any general rational matrix $G(s) \in \mathcal{R}^{p\times m}(s)$.

Our algorithm consists of the

- **factorization (4),**
- **eigen-factorizations(7) and (9) and**
- **solving the algebraic Riccati equation (13).**

Thus, the (J, J')-lossless factorization can be computed in a numerically reliable manner. A numerical example has also been given to illustrate the proposed algorithm.

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Other related current research topics in progress

- **Observer design (full order and reduced order) for nonlinear singular system.**
- **Intelligent Control using wavelet neural network design and Fuzzy network design**
  Application to time-delay problem for Nonlinear adaptive backstepping control design(IEEE NN).
- **Stability of Neural Networks with time-delays using LMI.**
  (Joint work with Prof Cao J)
- **Numerical Algorithms for Control problems**
  Computation of (J, J')-lossless factorization for general rational matrix (SIAM).
- **New stability issues on Ito singular systems**
  A special class of Stochastic control and filtering problems.
- **Missing data in Control and Network performance.**

Recent works in Singular systems

- Guoping Lu and Daniel W. C. Ho,“Generalized quadratic stability for continuous-time singular systems with nonlinear perturbation”, IEEE Trans. on Autom Control, accepted, 2006.
- Guoping Lu and Daniel W. C. Ho, “Full-order and reduced-order for Lipschitz descriptor systems: the unified LMI approach”, IEEE System and Circuits (II), accepted, 2006.
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