Dimension and measure theories for limsup sets defined by rectangles

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Outline

I. Metric Diophantine approximation

Metric Diophantine Approximation

- Motivation : Density of rational numbers.

Theorem

Rational numbers are dense in $\mathbb{R}$. Or equivalently, for any $x \in \mathbb{R}$,

$$|x - p/q| < \epsilon, \quad \text{for infinitely many } p/q.$$

Remark : This is a qualitative result in nature with no quantitative information.

- Main concerns : How well a real number can be approximated by rationals in quantitative sense.

II. Dirichlet & Minkowski

Dirichlet's theorem

Theorem (Dirichlet's Thm)

For any $(x_1, \cdots, x_d) \in \mathbb{R}^d$ *and any* $Q > 1$, *there exists integer* $q \leq Q$ *such that*

$$\max_{1 \leq i \leq d} \|q x_i\| < Q^{-1/d}.$$

Consequently,

$$\mathbf{x} \in \prod_{i=1}^{d} B\left(p_i/q, q^{-1-1/d}\right), \text{ i.m. } (p_1, \cdots, p_d, q) \in \mathbb{Z}^{d+1}.$$

Minkowski's theorem

Theorem (Minkowski's thm)

$t_1 + \cdots + t_d = 1$ *and positive. For any* $(x_1, \cdots, x_d) \in \mathbb{R}^d$ *and* $Q \in \mathbb{N}$, *there exists integer* $q \leq Q$ *such that*

$$\|qx_1\| < Q^{-t_1}, \cdots, \|qx_d\| < Q^{-t_d}.$$

So, consequently,

$$\mathbf{x} \in \prod_{i=1}^{d} B\Big(p_i/q, q^{-1-t_i}\Big), \text{ i.m. } (p_1, \cdots, p_d, q) \in \mathbb{Z}^{d+1}.$$

Improvability of Dirichlet and Minkowski ?

- Would be any improvability of Dirichlet and Minkowski ?

Almost impossible ! ! !

- Main reason :

They concern the property of ALL numbers.

▶ However, different numbers can be approximated by different degree.

eg. algebraic & Liouville.

▶ Thus, numbers are classified according to the degree they can be approximated.

limsup sets

- Classifications according to Dirichlet :

$$\mathcal{W}(\psi) := \left\{ \mathbf{x} \in [0,1]^d : \max_{1 \le i \le d} \|q x_i\| < \psi(q), \ \text{i.m. } q \in \mathbb{N} \right\}$$

$\longrightarrow$ We call them

limsup sets generated by balls.

- Classifications according to Minkowski :

$$\mathcal{R}(\Psi) := \left\{ \mathbf{x} \in [0,1]^d : \max_{1 \le i \le d} \|q x_i\| < \psi_i(q), \ \text{i.m. } q \in \mathbb{N} \right\}$$

$\longrightarrow$ We call them

limsup sets generated by rectangles.

Metric Diophantine approximation

- Main task : How about the size of limsup sets in the sense of

  measure & dimension.

- Two classes of limsup sets :
  - $\mathcal{W}(\psi)$:

    limsup sets defined by balls ;
  - $\mathcal{R}(\psi)$:

    limsup sets defined by rectangles ;

III. limsup sets for balls

I. Size in Lebesgue measure

Theorem (Khintchine's Thm, 1924)

Let $\psi : \mathbb{N} \to \mathbb{R}^+$ *be decreasing. Then*

$$\mathcal{L}(W(\psi)) = 0 \text{ or } 1 \Longleftrightarrow \sum_{q\geq 1} \psi(q) < \infty \text{ or } = \infty.$$

Theorem (Khintchine-Groshev's Thm, 1938)

Let $\psi : \mathbb{N} \to \mathbb{R}^+$ *be decreasing. Then*

$$\mathcal{L}(W_{mn}(\psi)) = 0 \text{ or } 1 \Longleftrightarrow \sum_{q\geq 1} q^{m-1}\psi^n(q) < \infty \text{ or } = \infty.$$

II. Size Hausdorff measure/dimension

Theorem (Jarník-Bescovitch's thm, 1929, 1934)

Let $\psi : \mathbb{N} \to \mathbb{R}^+$ *decreasing.*

$$\dim_H W(\psi) = \frac{2}{1+\tau}, \text{ where } \tau = \liminf_{q\to\infty} \frac{-\log\psi(q)}{\log q}.$$

Theorem (Jarník, 1931)

Let f *be a dimension function with* $f(r)/r$ *non-decreasing and* ψ *non-increasing. Then*

$$\mathcal{H}^f(W(\psi)) = \begin{cases} \infty, & \text{if } \sum_{q=1}^{\infty} qf\left(\frac{\psi(q)}{q}\right) = \infty\,; \\ 0, & \text{otherwise.} \end{cases}$$

Theorem (Bovey & Dodson, 1986, Dodson, 1992)

Let ψ non-increasing.

$$\dim_{\mathrm{H}} W_{mn}(\psi) = (m-1)n + \frac{m+n}{1+\tau}, \ \tau = \liminf_{q\to\infty} \frac{-\log\psi(q)}{\log q}.$$

Theorem (Dickinson & Velani, 1997)

Let f a dimension function with $f(r)/r^{mn}$ increasing to ∞ as $r \to 0$; $f(r)/r^{(m-1)n}$ increasing; ψ non-increasing. Then

$$\mathcal{H}^f(W_{mn}(\psi)) = \begin{cases} \infty, & \text{if } \sum_{q=1}^{\infty} q^{m+n-1} f\left(\frac{\psi(q)}{q}\right)\left(\frac{\psi(q)}{q}\right)^{-(m-1)n} = \infty; \\ 0, & \text{otherwise.} \end{cases}$$

Theorem (Levesley 1998, Bugeaud 2004)

Inhomogeneous case.

General principles

- Generally, one considers the size of the following limsup set

$$\Big\{x \in \mathbb{R}^d : x \in B(x_n, r_n),\ \text{i.m. } n \in \mathbb{N}\Big\}$$

or even more generally

$$\Big\{x \in \mathbb{R}^d : x \in \Delta(R_n, r_n),\ \text{i.m. } n \in \mathbb{N}\Big\} = \limsup_{n\to\infty} \Delta(x_n, r_n),$$

where R_n are subsets in $\mathbb{R}^d$ and Δ denotes the neighborhood.

Call them the *limsup set generated by balls*.

I : regular system

- I : an interval.
- Γ : be a countable set of real numbers in I
- $\mathcal{N} : \Gamma \to \mathbb{R}^+$ a positive function.

Definition (Baker & Schmidt, 1970)

The pair $(\Gamma, \mathcal{N})$ is called a regular system if $\exists c > 0$, s.t. for any interval $J \subset I$, for any $K \gg 1$, there are $\gamma_1, \cdots, \gamma_t$ in $\Gamma \cap J$ such that

$$\mathcal{N}(\gamma_j) \le K, \ |\gamma_i - \gamma_j| \ge K^{-1}, \ \ t \ge c|J|K.$$

Then consider

$$\Big\{x \in I : |x - \gamma| < \psi(\mathcal{N}(\gamma)), \text{ i.m. } \gamma \in \Gamma\Big\}.$$

II : Ubiquitous system

- Dodson, Rynne & Vickers (1990) :

  Ubiquitous system for balls in $\mathbb{R}^d$;

- Beresnevich, Dickinson & Velani (2006) :

  Ubiquity in abstract space.

- Ω : compact metric space with a δ-Ahlfors regular measure m.
- J : countable index set.
- $\{x_\alpha : \alpha \in J\}$: points in Ω.
- $\beta : J \to \mathbb{R}^+ : \alpha \to \beta_\alpha$: attaches a weight β_α to R_α.
- $\rho : \mathbb{R}^+ \to \mathbb{R}^+$: $\rho(r) \searrow 0$ as $r \to \infty$.

Definition (Local ubiquity system)

Call $(\{x_\alpha\}_{\alpha\in J}, \beta)$ a local m-ubiquity system with respect to ρ if there exists a constant $c>0$ such that for any ball B in X,

$$m\left(B\cap \bigcup_{\alpha\in J:\ell_n\le\beta_\alpha\le u_n} B(x_\alpha,\rho(u_n))\right)\ge cm(B), \quad \text{for } n\ge n_o(B). \quad (1)$$

- Consider the set

$$\Lambda(\psi)=\Big\{x\in\Omega : x\in B(x_\alpha,\psi(\beta_\alpha)),\ \text{i.m. } \alpha\in J\Big\}$$

where ψ is a non-increasing positive function defined on $\mathbb{R}^+$.

Theorem (Beresnevich, Dickinson & Velani, 2006)

(1). Assume the measure m *satisfying :* δ*-Alhfors regularity.*

(2). Suppose that $(\{x_\alpha\}_{\alpha\in J}, \beta)$ *is a local* m*-ubiquity system with respect to* ρ*.*

(3). Let f *be a dimension function such that* $f(r)/r^\delta$ *decreases.*

Let h *be given by*

$$h(u) := f(\psi(u))\rho(u)^{-\delta}.$$

- *(i) If* $\limsup h(u_n) = 0$ *and* $\rho(u_{n+1}) \le c\rho(u_n)$ *for all* $n \gg 1$*. Then*

$$\mathcal{H}^f(\Lambda(\psi)) = \infty, \quad \text{if } \sum_{n=1}^{\infty} h(u_n) = \infty.$$

- *(ii) If* $0 < \limsup h(u_n) \le \infty$*. Then* $\mathcal{H}^f(\Lambda(\psi)) = \infty$*.*

III : Measure transference principle

Measure transference principle : for limsup sets of balls,

Lebesgue measure statement $\Longrightarrow$ Hausdorff measure statement.

Theorem (Beresnevich & Velani, 2006)

Let $x_n \in \mathbb{R}^d$ and $r_n > 0$ $(n \in \mathbb{N})$. Let f be a dimension function with $f(r)/r^d$ nondecreasing as $r \to 0$. If

$$\Big\{x \in \mathbb{R}^d : x \in B(x_n, f^{1/d}(r_n)),\ \textit{i.m. } n \in \mathbb{N}\Big\}$$

is of full Lebesgue measure, then for any ball B in $\mathbb{R}^d$,

$$\mathcal{H}^f\Big(\big\{x \in B : x \in B(x_n, r_n),\ \textit{i.m. } n \in \mathbb{N}\big\}\Big) = \mathcal{H}^f(B).$$

- S. Jaffard (1996) :
  Dimension transference principle ;
- J. Barral & S. Seuret (2004, 2007)
  Conditional ubiquity & heterogeneous ubiquity ;
- A. Fan, J. Schmeling & S. Troubetzkoy (2013) :
  Multifractal mass transference principle.
- D. Feng, E. Järvenpää, M. Järvenpää, V. Suomala (2018) :
  Random limsup sets ;
- H. Koivusalo & M. Rams (2018) :
  Mass transference principle from balls to open sets ;
- D. Allen & V. Beresnevich (2018) :
  Mass transference principle for linear forms (isotropic thicken) ;
- D. Allen & S. Baker (2019) :
  Mass transference principle for general forms ;
- F. Ekström, E. Järvenpää, M. Järvenpää (2020) :
  limsup sets of rectangles in the Heisenberg group ;
- D. Édouard (2023) :
  Mass transference principle for quasi-Bernoulli measures.
- ⋮

IV. limsup sets for rectangles

Aims at measure the size of the sets

- Simultaneous

$$W_{1n}(\Psi) := \{(x_1, \cdots, x_n) \in \mathbb{R}^n : \|qx_i\| < \psi_i(q),\ 1 \le i \le n,\ \text{i.m. } q \in \mathbb{N}\}.$$

- More generally, the linear form :

$$W_{mn}(\Psi) = \Big\{x \in \mathbb{R}^{m\times n} : \|q_1x_{1i} + q_2x_{2i} + \cdots + q_mx_{mi}\| < \psi_i(q), \\ 1 \le i \le n,\ \ \text{i.m. } q = (q_1, \cdots, q_m) \in \mathbb{Z}^m\Big\}.$$

- We call these sets are

limsup sets generated by rectangles.

An related question

- Multiplicative Diophantine approximation : consider the size of

$$M(t) := \Big\{(x_1, \cdots, x_d) \in [0,1)^d : \|qx_1\| \cdots \|qx_d\| < \psi(q)^t, \text{ i.m. } q \in \mathbb{N}\Big\}.$$

If define

$$M(t_1, \cdots, t_d) = \Big\{x \in [0,1)^d : \|qx_i\| < \psi(q)^{t_i}, \ 1 \le i \le d, \text{ i.m. } q \in \mathbb{N}\Big\}$$

then for any integer N large,

$$\bigcup_{(j_1,\cdots,j_d)\in\mathbb{N}^d_{\ge 0}: j_1+\cdots+j_d=N} M\Big(\frac{j_1 t}{N}, \cdots, \frac{j_d t}{N}\Big) \subset M(t)$$

$$\subset \bigcup_{(i_1,\cdots,i_d)\in\mathbb{N}^d_{\ge 0}: j_1+\cdots+j_d=N-d} M\Big(\frac{j_1 t}{N}, \cdots, \frac{j_d t}{N}\Big).$$

Measure and dimension

Theorem (Khintchine 1926, Schmidt 1960, Gallagher 1962, Sprindzuk 1970)

Assume that ψ_i is non-increasing for all $1 \le i \le n$. Then

$$\mathcal{L}^d(\mathcal{W}_{mn}(\Psi)) = 0, \text{ or full } \iff \sum_{q=1}^{\infty} q^{m-1}\psi_1(q)\cdots\psi_n(q) < \infty, \text{ or } = \infty.$$

Theorem (Rynne 1998, Dickinson & Rynne 2000)

Let $\psi_i(q) = q^{-\tau_i}$ for each $1 \le i \le n$. Assume that $\tau_1 \le \tau_2 \le \cdots \le \tau_n$ and $\sum_{i=1}^n \tau_i \ge m$. Then

$$\dim_{\mathrm{H}} \mathcal{W}_{mn}(\Psi) = m(n-1) + \min_{1\le i\le n} \frac{n+m+i\tau_i - \sum_{j\le i}\tau_j}{1+\tau_i}.$$

- Wrong claim for general Ψ.

Questions

- What is the right dimension for general $(\psi_1, \cdots, \psi_n)$?
- Is there any general principle for limsup set defined by rectangles ?
  - Measure theory ?
  - Dimension theory ?

IV. Main result.

The setting

- J : countable index set.
- X_i : compact metric space, $1 \le i \le d$.
- μ_i : δ_i-Ahlfors regular measure on X_i.
- $x_{\alpha,i}$: point of X_i.
- $\rho : \mathbb{N} \to \mathbb{R}^+$ decreasing, called ubiquitous function.

Define

$$(X,\mu) = \Big(\prod_{i=1}^{d} X_i, \prod_{i=1}^{d} \mu_i\Big); \quad \mathbf{x}_\alpha = \prod_{i=1}^{d} x_{\alpha,i}, \quad \alpha \in J.$$

$$B(\mathbf{x}_\alpha, \mathbf{r}) = \prod_{i=1}^{d} B(x_{\alpha,i}, r_i).$$

I (1). Dimension under ubiquity

Motivated by Minkowski's theorem, we introduce *ubiquity for rectangles*.

Definition (μ-local ubiquity for rectangles)

$\exists\ c > 0$ s.t. for any ball $B(x, r)$, $\exists\ n_0(B)$ s.t. for all $n \geq n_0(B)$

$$\mu\Big(B \cap \bigcup_{\alpha:\ell_n \leq \beta_\alpha \leq u_n} \prod_{i=1}^{d} B(x_{\alpha,i}, \rho(u_n)^{a_i})\Big) \geq c \cdot \mu(B).$$

- Consider the limsup set defined by a sequence of rectangles :

$$W(\mathbf{t}) = \Big\{x \in X : x \in \prod_{i=1}^{d} B(x_{\alpha,i}, \psi_i(\beta_\alpha)),\ \text{i.m. } \alpha \in J\Big\}.$$

An simplest case : let $\psi_i(r) = \rho(r)^{a_i+t_i}$, for $1 \le i \le d$.

Theorem (W-Wu, Math. Ann., 2021)

Assume μ_i is δ_i-Ahlfors. Under the hypothesis of μ-local ubiquity for rectangles, one has

- $$\dim_{\mathrm{H}} W(\mathbf{t}) \ge \min_{A_i \in \mathcal{A}} \left\{ \sum_{k \in \mathcal{K}_1} \delta_k + \sum_{k \in \mathcal{K}_2} \delta_k + \frac{\sum_{k \in \mathcal{K}_3} a_k \delta_k - \sum_{k \in \mathcal{K}_2} t_k \delta_k}{A_i} \right\}$$

where

$$\mathcal{A} = \{a_i, a_i + t_i, 1 \le i \le d\}$$

and $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3$ are defined as follows

$$\mathcal{K}_1 = \Big\{k : A_i < a_k\Big\}, \quad \mathcal{K}_2 = \Big\{k : a_k + t_k \le A_i\Big\}, \quad \mathcal{K}_3 = [1, d] \setminus (\mathcal{K}_1 \cup \mathcal{K}_2).$$

- *Moreover, for any ball B in X,*

$$\mathcal{H}^s(B \cap W(\mathbf{t})) = \mathcal{H}^s(B).$$

I (2) : Dimension under full measure

Theorem (W-Wu, Math. Ann., 2021)

Assume μ_i is δ_i-Ahlfors. Under the full measure property that

$$\mu\left(\limsup_{\alpha:\beta_\alpha\to\infty} B(x_\alpha, \rho(\beta_\alpha)^{\mathbf{a}})\right) = \mu(X),$$

one has the same dimensional result for $W(\mathbf{t})$ as above.

How to cover one rectangle

Let

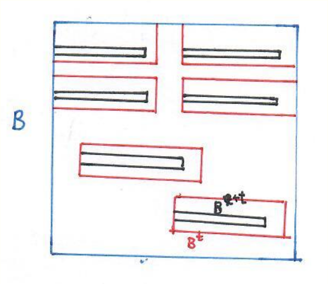
$$R = \prod_{i=1}^{d} B(x_i, r_i).$$

- Falconer's singular function :

$$\varphi^s(R) = \min\Big\{ \prod_{k: r_k \geq r_i} \frac{r_k}{r_i} \cdot r_i^s : 1 \leq i \leq d \Big\}.$$

In other words, cover R by balls of radius being the sidelengths of R

How to cover a collection of rectangles $\mathcal{R}_n$



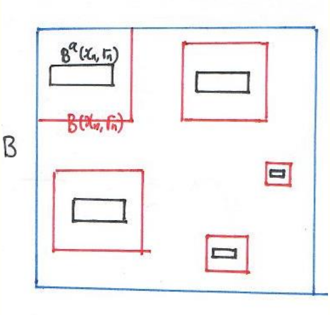
Red : big rectangles ; Black : shrunk rectangles

$$\prod_{i=1}^{d} B(x_i, r^{a_i}), \qquad \prod_{i=1}^{d} B(x_i, r^{a_i+t_i}).$$

For example in $\mathbb{R}^3$ assume

$$r^{a_3+t_3} \le r^{a_3} \le r^{a_2+t_2} \le r^{a_2} \le r^{a_1+t_1} \le r^{a_1}.$$

Local part : transference from balls to rectangles



Red : big balls ;   Black : shrunk rectangles

II : Measure theory

Definition (μ-local ubiquity for rectangles)

$\exists\ c>0$ s.t. for any ball $B(x,r)$, $\exists\ n_0(B)$ s.t. for all $n \geq n_0(B)$

$$\mu\Big(B \cap \bigcup_{\alpha:\ell_n \leq \beta_\alpha \leq u_n} \prod_{i=1}^{d} B(x_{\alpha,i}, \rho_i(u_n))\Big) \geq c \cdot \mu(B).$$

- Consider the limsup set defined by a sequence of rectangles :

$$W(\Psi) = \Big\{x \in X : x \in \prod_{i=1}^{d} B(x_{\alpha,i}, \psi_i(\beta_\alpha)), \text{ i.o. } \alpha \in J\Big\}.$$

Theorem (Kleinbock-W, Adv. Math., 2023)

(1). δ_i*-Ahlfors regularity for* μ_i*,*

(2). Ubiquity for rectangles.

(3). Ψ *decreasing, either* Ψ *or* ρ *is* λ*-regular.*

Then

$$\mu\big(\mathcal{W}(\Psi)\big) = \mu(X) \quad \textit{if} \quad \sum_{n\geq 1}\prod_{i=1}^{d}\left(\frac{\psi_i(u_n)}{\rho_i(u_n)}\right)^{\delta_i} = \infty.$$

λ-regular : $\rho(u_{n+1}) \leq \lambda \cdot \rho(u_n)$ for some $\lambda < 1$.

An example

$$\Big\{(x,y): \|2^n x\| < \psi_1(n), \|3^n y\| < \psi_2(n), \text{ for infinitely many } n \in \mathbb{N}\Big\}.$$

- Resonant sets :

$$x_\alpha = \left(\frac{a}{2^n}, \frac{b}{3^n}\right), 0 \le a < 2^n, 0 \le b < 3^n, n \ge 1$$

- Ubiquitous function

$$\rho(n) = (2^{-n}, 3^{-n})$$

since

$$[0,1]^2 = \bigcup_{0 \le a < 2^n, 0 \le b < 3^n} B\left(\frac{a}{2^n}, \frac{1}{2^n}\right) \times B\left(\frac{b}{3^n}, \frac{1}{3^n}\right).$$

Application 1

$$W(\Psi) = \Big\{x \in [0,1]^n : \|qx_i\| < \psi_i(q),\ 1 \le i \le n, \ \text{ i.m. } q \in \mathbb{N}\Big\}$$

$$W(\tau) = \Big\{x \in [0,1]^n : \|qx_i\| < q^{-\tau_i},\ 1 \le i \le n, \ \text{ i.m. } q \in \mathbb{N}\Big\}$$

Theorem

Let ψ_i decreasing for all $1 \le i \le n$.

$$\dim_{\mathrm{H}} W(\tau) = \min\left\{\frac{n+1+\sum_{k=i}^{n}(\tau_i - \tau_k)}{1+\tau_i}, 1 \le i \le n\right\} := s(\tau)$$

$$\dim_{\mathrm{H}} W(\Psi) = \sup\Big\{s(\tau) : \tau \in \mathcal{U}\Big\},$$

where $\mathcal{U}$ is the set of the accumulation points of

$$\left\{\Big(\frac{-\log\psi_1(q)}{\log q}, \cdots, \frac{-\log\psi_n(q)}{\log q}\Big) : q \in \mathbb{N}\right\}.$$

Application 2 : linear form

$$W_{mn}(\psi) = \Big\{x \in \mathbb{R}^{mn} : \|q_1 x_{1i} + \cdots + q_m x_{mi}\| < \psi_i(q)q,\ 1 \le i \le n,\ \text{i.m. } q \in \mathbb{N}$$

$$W_{mn}(\tau) = \Big\{x \in \mathbb{R}^{mn} : \|q_1 x_{1i} + \cdots + q_m x_{mi}\| < q^{-\tau_i} q,\ 1 \le i \le n,\ \text{i.m. } q \in \mathbb{N}$$

Theorem

Let ψ_i decreasing for all $1 \le i \le n$.

$$\dim_{\mathrm{H}} W_{mn}(\tau) = \min_{1 \le i \le m} \left\{(m-1)n + \frac{m + n + \sum_{k=i}^{m}(\tau_i - \tau_k)}{\tau_i}\right\} := s(\tau)$$

$$\dim_{\mathrm{H}} W_{mn}(\Psi) = \sup\Big\{s(\tau) : \tau \in \mathcal{U}\Big\}.$$

Other applications

- 3. Shrinking target problem :

$$S(\psi) := \Big\{(x_1, \cdots, x_d) \in \prod_{i=1}^{d} \mathcal{C}_i : \|b_i^n x_i - x_{o,i}\| < \psi_i(n), \text{ i.m. } n \in \mathbb{N}\Big\}.$$

- 4. Multiplicative Diophantine :

$$M_c(\psi) := \Big\{(x, y) \in \mathcal{C}_a \times \mathcal{C}_b : \|a^n x - x_o\| \cdot \|b^n y - y_o\| < \psi(n), \text{ i.m. } n \in \mathbb{N}\Big\}.$$

subsequent works

- Beresnevich, Levesley & Ward (Math. Proc. Camb. Philos. 2023),
  Diophantine approximation on p-adic fields.
- Eceizabarrena & Ponce-Vanegas (Commun. Pure Appl. Anal., 2022),
  convergence of the solutions of the Schrödinger equation ;
- Eceizabarrena & Ponce-Vanegas (arXiv : 2108.10339),
  pointwise convergence of the solutions of dispersive equations ;
- Li, Liao, Velani & Zorin (Adv. Math., 2023),
  shrinking target problems on torus.
- D. Allen & Wang, (arXiv :2103.06822)
  Diophantine approximation on manifolds.
- Robert, Hussain, Shulga & Ward, (arXiv : 2308.16603)
  p-adic field, complex number field, formal Laurent series.

Thanks for your attention !