Almost sure dimensional properties for the spectrum and the density of states of Sturmian Hamiltonians

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Fractal geometry and related topics

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Discrete Schrödinger operator

Given $V : \mathbb{Z} \to \mathbb{R}$ bounded. Define the discrete Schrödinger operators $H_V : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ as

$$\begin{aligned} H_V\psi &:= \Delta\psi + V\psi \\ (H_V\psi)_n &:= (\psi_{n+1} + \psi_{n-1}) + V_n\psi_n. \end{aligned}$$

Fact: H_V is bounded, self-adjoint, the spectrum $\sigma(H_V) \subset \mathbb{R}$ is compact.
Physically: It describe the motion of an electron in a material. The spectral property is related to the conductivity of the material.

Spectral measure

For any $\psi \in \ell^2(\mathbb{Z})$, the spectral measure μ_ψ is defined by (via Riesz presentation theorem)

$$\int_{\sigma(H_V)} f(E) d\mu_\psi(E) := \langle \psi, f(H_V)\psi \rangle, \quad f \in C(\sigma(H_V)).$$

Define the spectral measure of H_V as

$$\mu_V := \frac{\mu_{\delta_0} + \mu_{\delta_1}}{2}.$$

Fact: For any $\psi \in \ell^2(\mathbb{Z})$, one has $\mu_\psi \ll \mu_V$.
Physically: If μ_V is a.c. (p.p., "s.c.") then the material is a conductor (insulator, "semi-conductor")

Periodic potential case— Floquet-Bloch theory

Theorem (Floquet-Bloch)

Assume V is n-periodic, then the spectrum of H_V is given by

$$\sigma(H_V) = \{E \in \mathbb{R} : |t_V(E)| \leq 2\} = B_1 \cup B_2 \cup \cdots \cup B_n,$$

where t_V is a polynomial of degree n, called the trace polynomial of H_V. The spectral measure $\mu_V \ll \mathcal{L}|_{\sigma(H_V)}$.

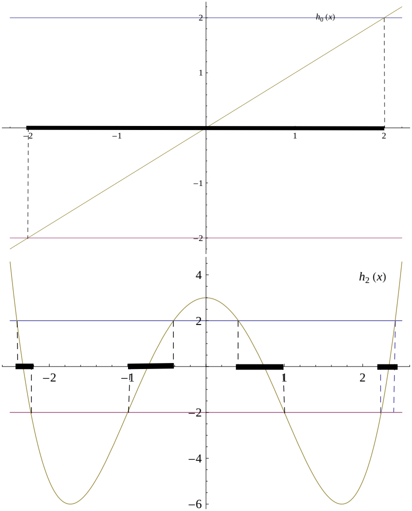
- For $V \equiv 0$. We have

$$t_0(E) = E; \quad \sigma(H_0) = [-2, 2]; \quad \mu_0 = \frac{\chi_{[-2,2]}(E)dE}{\pi\sqrt{4 - E^2}}$$

- V is 4-periodic and

$$V|_{[1,4]} = (1, -1, -1, 1); \quad t(E) = E^4 - 6E^2 + 3.$$

Pictures of the Spectra



Quasi-periodic potentials

Two classes of quasi-periodic potentials are heavily studied, they all have the following form:

$$V_{f,\alpha,\lambda,\theta}(n) = \lambda f(\theta + n\alpha) \tag{1}$$

where $f : \mathbb{S}^1 \to \mathbb{R}$ is bounded, $\alpha \in [0,1] \setminus \mathbb{Q}$, $\lambda > 0$ and $\theta \in \mathbb{S}^1$.

- Almost Mathieu potential:

  $$f(x) = 2cos2\pi x.$$

  The related operator is called AMO.

- Sturmian potential:

  $$f(x) = \chi_{[1-\alpha,1)}(x).$$

  The related operator is called Sturmian Hamiltonian.

Spectrum and density of states

For operator with potential (1), by the general theory of ergodic Schrödinger operators, the spectrum is independent of θ. So we write

$$\Sigma^f_{\alpha,\lambda} := \sigma(H_{V_{f,\alpha,\lambda,\theta}}).$$

Another important measure, called density of states (DOS) of the operator, is defined as the average of the spectral measures:

$$\mathcal{N}^f_{\alpha,\lambda} := \int_{\mathbb{S}^1} \mu_{V_{f,\alpha,\lambda,\theta}} d\theta.$$

Theorem (B. Simon 2007)

$\mathcal{N}^f_{\alpha,\lambda}$ is the harmonic measure on $\Sigma^f_{\alpha,\lambda}$.

Cantor spectrum–fractal is coming

To study quasi-periodic operators, we do the periodic approximation: Choose potentials $V^{(n)}$ which is k_n-periodic such that $V^{(n)} \to V$ in suitable sense. Then $H_n := H_{V^{(n)}} \stackrel{s}{\to} H_V$. As a consequence,

$$d_H(\sigma(H_n), \sigma(H_V)) \to 0.$$

By Floquet-Bloch theory, $\sigma(H_n)$ is made of k_n non-overlapping bands. When $n \to \infty$, the spectrum has the tendency to be a Cantor set.

The spectrum of AMO —$\lambda = 1; \alpha = [0; 2, 4, 8, 16, \cdots]$

$$\alpha_1 = \frac{1}{2}; \quad \alpha_2 = \frac{4}{9}; \quad \alpha_3 = \frac{33}{74}; \quad \alpha_1 = \frac{532}{1193}.$$

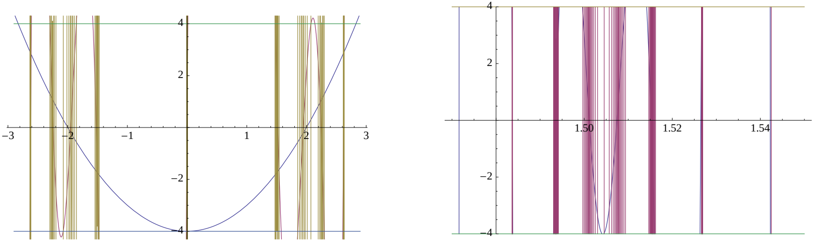


Figure: Left– t_1, t_2, t_3 ; right– t_3, t_4

For the spectrum of AMO, the following holds:

Theorem ($\cdots$,Avila-Jitomirskaya(2009); $\cdots$,Avila-Krikorian(2006))

The spectrum $\Sigma_{\alpha,\lambda}^{AMO}$ is a Cantor set of Lebesgue measure $|4-4\lambda|$.

For the DOS of AMO, the following holds:

Theorem (Avila-Damanik(2008))

If $\lambda \neq 1$, then the DOS $\mathcal{N}_{\alpha,\lambda}^{AMO}$ is a.c..

For the spectrum of Sturmian Hamiltonian, the following holds:

Theorem (Bellissard-Iochum-Scoppola-Testart(1989))

The spectrum $\Sigma_{\alpha,\lambda}^{SH}$ of Sturmian Hamiltonian is a Cantor set of Lebesgue measure zero.

Deterministic results

Now we focus on Sturmian Hamiltonian and simply the notions to

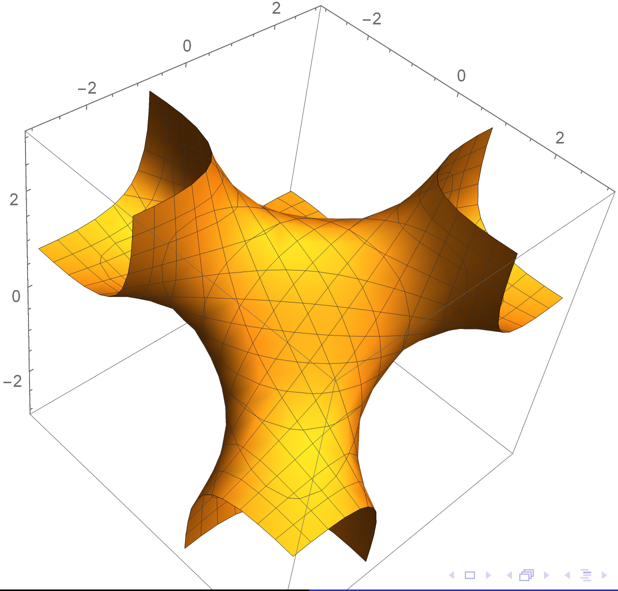
$$H_{\alpha,\lambda,\theta}, \quad \Sigma_{\alpha,\lambda}, \quad \mathcal{N}_{\alpha,\lambda}.$$

• Fibonacci Hamiltonian:The operator $H_{\alpha_1,\lambda,\theta}$ with golden ratio $\alpha_1 := (\sqrt{5}+1)/2$. This model was introduced by Kohmoto et. al. and Ostlund et. al.(1983) as a model for quasicrystal.
Define the Fibonacci trace map $\mathbf{T} : \mathbb{R}^3 \to \mathbb{R}^3$ as

$$\mathbf{T}(x, y, z) := (2xy - z, x, y).$$

Then $G(x, y, z) := x^2 + y^2 + z^2 - 2xyz - 1$ is invariant under $\mathbf{T}$.
So for $\lambda > 0$, $\mathbf{T}$ preserves the cubic surface

$$S_\lambda := \{(x, y, z) \in \mathbb{R}^3 : G(x, y, z) = \lambda^2/4\}.$$


2
0
−2
−2
0
2
2
0
−2

Fibonacci Hamiltonian

Write $\mathbf{T}_\lambda := \mathbf{T}|_{S_\lambda}$ and let Λ_λ be the attractor of $\mathbf{T}_\lambda$. Then Λ_λ is a locally maximal compact transitive hyperbolic set of $\mathbf{T}_\lambda$

Theorem (Casdagli 1986,$\cdots$,Damanik-Gorodetski-Yessen 2016)

For Fibonacci Hamiltonian, the following hold:
1) The spectrum $\Sigma_{\alpha_1,\lambda}$ satisfies

$$\dim_H \Sigma_{\alpha_1,\lambda} = \dim_B \Sigma_{\alpha_1,\lambda} =: D(\alpha_1,\lambda). \tag{2}$$

2) $D(\alpha_1,\lambda)$ satisfies Bowen's formula: $D(\alpha_1,\lambda)$ solves the equation $P(t\phi_\lambda)=0$, where ϕ_λ is the geometric potential on Λ_λ

$$\phi_\lambda(x) := -\log \|D\mathbf{T}_\lambda(x)|_{E^u}\|.$$

Theorem

3) The function $D(\alpha_1,\cdot)$ is analytic on $(0,\infty)$ and

$$\lim_{\lambda\to 0} D(\alpha_1,\lambda)=1;\quad \lim_{\lambda\to\infty} D(\alpha_1,\lambda)\log\lambda=\log(1+\sqrt{2}). \tag{3}$$

4) The DOS $\mathcal{N}_{\alpha_1,\lambda}$ is exact-dimensional and consequently

$$\dim_H \mathcal{N}_{\alpha_1,\lambda}=\dim_P \mathcal{N}_{\alpha_1,\lambda}=: d(\alpha_1,\lambda). \tag{4}$$

5) $d(\alpha_1,\lambda)$ satisfies Ledrappier-Young's formula:

$$d(\alpha_1,\lambda)=\dim_H \mu_{\lambda,\max}=\frac{\log\alpha_1}{\mathrm{Lyap}^u\mu_{\lambda,\max}}, \tag{5}$$

where $\mu_{\lambda,\max}$ is the measure of maximal entropy of $\mathbf{T}_\lambda$, and $\log\alpha_1, \mathrm{Lyap}^u\mu_{\lambda,\max}$ are the entropy and the unstable Lyapunov exponent of $\mu_{\lambda,\max}$, respectively.

Theorem

6) The function $d(\alpha_1, \cdot)$ *is analytic on* $(0, \infty)$ *and*

$$\lim_{\lambda \to 0} d(\alpha_1, \lambda) = 1; \quad \lim_{\lambda \to \infty} d(\alpha_1, \lambda) \log \lambda = \frac{5+\sqrt{5}}{4} \log \alpha_1. \tag{6}$$

Sturmian Hamiltonian

Assume $\alpha \in [0,1] \setminus \mathbb{Q}$ has expansion $\alpha = [0; a_1, a_2, \cdots]$. Define

$$K_*(\alpha) = \liminf_{n\to\infty} \left(\prod_{j=1}^{n} a_j\right)^{1/n}; \quad K^*(\alpha) = \limsup_{n\to\infty} \left(\prod_{j=1}^{n} a_j\right)^{1/n}.$$

Theorem (Liu-Wen(2004),$\cdots$,Liu-Q-Wen(2014))

Assume $\lambda \geq 24$. then
1) The following dichotomies hold:

$$\begin{cases} \dim_H \Sigma_{\alpha,\lambda} \in (0,1) & \text{if} \quad K_*(\alpha) < \infty \\ \dim_H \Sigma_{\alpha,\lambda} = 1 & \text{if} \quad K_*(\alpha) = \infty \end{cases}$$

$$\begin{cases} \overline{\dim}_B \Sigma_{\alpha,\lambda} \in (0,1) & \text{if} \quad K^*(\alpha) < \infty \\ \overline{\dim}_B \Sigma_{\alpha,\lambda} = 1 & \text{if} \quad K^*(\alpha) = \infty \end{cases}.$$

Sturmian Hamiltonian

Theorem (continued)

2) $\underline{D}(\alpha,\cdot)$ and $\overline{D}(\alpha,\cdot)$ are Lipschitz continuous on any bounded interval of $[24,\infty)$, where

$$\underline{D}(\alpha,\lambda) := \dim_H \Sigma_{\alpha,\lambda} \quad \text{and} \quad \overline{D}(\alpha,\lambda) := \overline{\dim}_B \Sigma_{\alpha,\lambda}.$$

3) There exist two constants $0 < \rho_*(\alpha) \le \rho^*(\alpha) \le \infty$ such that

$$\lim_{\lambda\to\infty} \underline{D}(\alpha,\lambda)\log\lambda = \rho_*(\alpha) \quad \text{and} \quad \lim_{\lambda\to\infty} \overline{D}(\alpha,\lambda)\log\lambda = \rho^*(\alpha).$$

All of above results are based on a very explicit coding of the spectrum established by Raymond:

The coding of the spectra

Theorem (Raymond 1997 (Preprint))

For any $\lambda > 4$ and $\alpha \in [0,1] \setminus \mathbb{Q}$, there exists a symbolic space Ω_α and a coding map $\pi_\alpha : \Omega_\alpha \to \Sigma_{\lambda,\alpha}$.

For Fibonacci Hamiltonian, the symbolic space Ω_{α_1} is essentially the subshift of finite type with alphabet $\mathcal{A} := \{e_1, e_2, e_3, e_4\}$ and coincidence matrix

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

In general, Ω_α is a subshift defined by sequences of alphabets $\{\mathcal{A}_{a_n} : n \geq 0\}$ and incidence matrices $\{A_{a_n a_{n+1}} : n \geq 0\}$.

Bellissard's conjecture and Damanik-Gorodetski's result

Until now, all the results are stated for deterministic frequencies. How about the dimensional properties of $\Sigma_{\alpha,\lambda}$ and $\mathcal{N}_{\alpha,\lambda}$ for Leb. typical frequency?
Bellissard had the following conjecture in 1980s:

Conjecture(Bellissard 1980s): For every $\lambda > 0$, the Hausdorff dimension of $\Sigma_{\alpha,\lambda}$ is Leb. a.e. constant in α.

Theorem (Damanik-Gorodetski 2015)

For every $\lambda \geq 24$, there exists two numbers $0 < \underline{D}(\lambda) \leq \overline{D}(\lambda)$ such that for Lebesgue almost every $\alpha \in [0,1] \setminus \mathbb{Q}$,

$$\dim_H \Sigma_{\alpha,\lambda} = \underline{D}(\lambda) \quad \textit{and} \quad \overline{\dim}_B \Sigma_{\alpha,\lambda} = \overline{D}(\lambda).$$

Idea of the proof(Based on Liu-Q-Wen 2014): Show that $\underline{D}(\cdot, \lambda)$ is measurable and invariant under Gauss measure G. Then use the ergodicity of G. The same for $\overline{D}(\cdot, \lambda)$.

Natural questions: for fixed $\lambda \geq 24$, whether $\underline{D}(\lambda) = \overline{D}(\lambda)$ holds? Does the full measure set of frequencies depend on λ? How regular are the functions $\underline{D}(\lambda)$ and $\overline{D}(\lambda)$? What can one say about the DOS? etc.

a.s. dimensional property of the spectrum

For the spectrum, we have

Theorem (Cao-Q 2023)

There exist a subset $\tilde{\mathbb{I}} \subset [0,1] \setminus \mathbb{Q}$ *of full Lebesgue measure and a function* $D : [24, \infty) \to (0,1)$ *such that the following holds:*
1) For any $(\alpha, \lambda) \in \tilde{\mathbb{I}} \times [24, \infty)$, *the spectrum* $\Sigma_{\alpha,\lambda}$ *satisfies*

$$\dim_H \Sigma_{\alpha,\lambda} = \dim_B \Sigma_{\alpha,\lambda} = D(\lambda). \tag{7}$$

2) $D(\lambda)$ *satisfies a Bowen type formula:* $D(\lambda)$ *is the unique zero of a relativized pressure function* $P_{\mathbf{G}}(\Psi^*_{t,\lambda})$.
3) $D(\lambda)$ *is Lipschitz continuous on any bounded interval of* $[24, \infty)$ *and there exists a constant* $\rho \in (0,1)$ *such that*

$$\lim_{\lambda \to \infty} D(\lambda) \log \lambda = -\log \rho. \tag{8}$$

Remark

1) Our result improve D-G's thm in two aspects: firstly, the full measure set $\tilde{\mathbb{I}}$ is independent of λ. Secondly, our result shows that indeed $\underline{D}(\lambda) = \overline{D}(\lambda)$.
2) Item 2) is a random version of Bowen's formula. Similarly, (7) is a random version of (2) and (8) is a random version of (3).
3) We are inspired by Feng-Shu(2009).

a.s. dimensional property of the DOS

For the DOS, we have

Theorem (Cao-Q 2023)

There exist a subset $\hat{\mathbb{I}} \subset [0,1] \setminus \mathbb{Q}$ *of full Lebesgue measure and a function* $d : [24,\infty) \to (0,1)$ *such that the following hold:*
1) For any $(\alpha,\lambda) \in \hat{\mathbb{I}} \times [24,\infty)$, $\mathcal{N}_{\alpha,\lambda}$ *is exact-dim. and*

$$\dim_H \mathcal{N}_{\alpha,\lambda} = \dim_P \mathcal{N}_{\alpha,\lambda} = d(\lambda). \tag{9}$$

2) $d(\lambda)$ *satisfies a Ledrappier-Young type formula:*

$$d(\lambda) = \frac{\gamma}{-(\Psi_\lambda)_*(\mathscr{N})}, \tag{10}$$

where γ *is the Lévy's constant,* $\mathscr{N}$ *is a Gibbs measure on the*

Theorem (Cao-Q(2023)continued)

global symbolic space Ω, and Ψ_λ is the geometric potential on Ω.
3) $d(\lambda)$ is Lipschitz on $[24, \infty)$ and there exists $\varrho \in (0, 1)$ s.t.

$$\lim_{\lambda \to \infty} d(\lambda) \log \lambda = -\log \varrho. \tag{11}$$

Remark

1) $\mathcal{N}_{\alpha,\lambda}$ is kind of measure of maximal entropy with entropy γ.
2) (10) is a random version of (5), where γ is the entropy and $-(\Psi_\lambda)_*(\mathscr{N})$ is the Lyapunov exponent of $-\Psi_\lambda$ w.r.t. $\mathscr{N}$.
Similarly, (9) is a random version of (4), and (11) is a random version of (6).

By viewing the spectrum as kind of random attractor, we transfer the spectral problem to a dynamical problem. Then we combine the tools from the thermodynamical formalism of topological Markov chain over countable alphabet and random dynamical systems, derive the desired result.

Thanks for your attention!

Jie Cao and Yanhui Qu, *Almost sure dimensional properties for the spectrum and the density of states of Sturmian Hamiltonians*, Arxiv:2310.07305.