# Reflected diffusion on uniform domains

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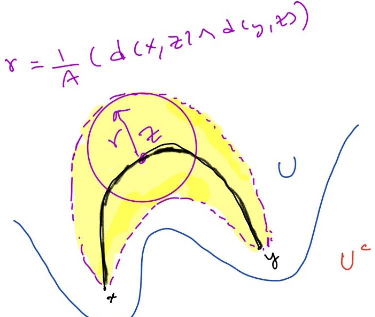
- Given a diffusion process on a space $X$ and a domain $U \subset X$, does the reflected diffusion on $U$ inherit properties from diffusion on the $X$?
- (Martio, Sarvas '79) A connected, non-empty, proper open set $U \subsetneq X$ is said to be a *A-uniform domain*($A \geq 1$) if for every pair of points $x, y \in U$, there exists a curve $\gamma$ in $U$ from $x$ to $y$ such that its diameter $\mathrm{diam}(\gamma) \leq Ad(x,y)$, and

$$\mathrm{dist}\,(z, U^c) \geq A^{-1} \min\left(d(x,z), d(y,z)\right) \quad \text{for all } z \in \gamma.$$



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- (Rajala '21) Uniform domains are abundant in the sense that every bounded domain can be approximated by a uniform domain.
- Given a complete, doubling metric space $(X, d)$ that is bi-Lipschitz equivalent to a length space, a bounded domain $\Omega$ and $\epsilon > 0$, there exist uniform domains $\Omega_o$ and $\Omega_i$ such that

$$\Omega_i \subset \Omega \subset \Omega_0, \quad \Omega_o \subset [\Omega]_\epsilon, \quad (\Omega_i)^c \subset [\Omega^c]_\epsilon.$$

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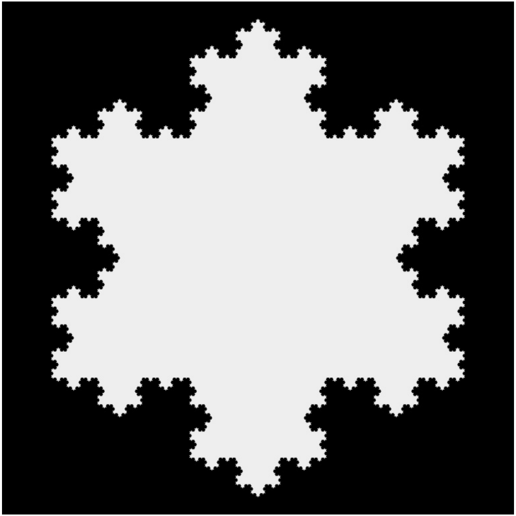
- Uniform domains can have fractal boundaries and are far from smooth in general.

# Koch snowflake is a uniform domain

Image: Wikipedia



# Reflected diffusions on $\mathbb{R}^n$

- (Skorohod '62) For a smooth domain $U$ in $\mathbb{R}^n$, the reflected Brownian motion on $U$ is the solution to the stochastic differential equation

$$Y(t) = Y(0) + B(t) + \int_0^t \vec{n}(Y(s))\, dL_s,$$

where $B(t)$ is the standard Brownian motion on $\mathbb{R}^n$, $L_s$ is the 'boundary local time' of the process $Y(s)$ and $\vec{n}(x)$ is the inward pointing unit normal vector at $x \in \partial U$.

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- (Fukushima '67) The Dirichlet form approach involves the bilinear form

$$\mathcal{E}_U(f,f) := \frac{1}{2}\int_U |\nabla f|^2(x)\, dx,$$

for all $f \in W^{1,2}(U)$. This defines a Markov process on $U^*$ where $U^*$ is an abtract closure of $U$ (Martin-Kuramochi compactification).

# Symmetric Markov process

- $(X, d)$ complete, locally compact metric space equipped with a Radon measure $m$.
- Let $(B_t)_{t\geq 0}$ be a $m$-symmetric Markov process. That is the Markov semigroup $(P_t)_{t\geq 0}$

$$P_t f(x) := \mathbb{E}_x f(B_t) = \mathbb{E}[f(B_t)|B_0 = x],$$

satisfies

$$\langle P_t f, g \rangle = \langle f, P_t g \rangle, \quad \text{for all } f, g \in L^2(X, m)$$

- Example: Brownian motion on $\mathbb{R}^n$ is symmetric with respect to the Lebesgue measure

# Dirichlet form associated with a symmetric diffusion

- The corresponding Dirichlet form $(\mathcal{E}, \mathcal{F})$ is defined by

$$\mathcal{E}(f, f) = \lim_{t \downarrow 0} \frac{1}{t} \langle f, (I - P_t) f \rangle, \text{for all } f \in \mathcal{F},$$

where

$$\mathcal{F} = \left\{ f \in L^2(X, m) : \lim_{t \downarrow 0} \frac{1}{t} \langle f, (I - P_t) f \rangle < \infty \right\}$$

- For the standard Brownian motion on $\mathbb{R}^n$ ($m =$Lebesgue)

$$\mathcal{E}(f, f) = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla f|^2 \, dm, \quad \mathcal{F} = W^{1,2}(\mathbb{R}^n).$$

# Dirichet form

- Let $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form on $L^2(X, m)$.
- $\mathcal{F}$ is a dense linear subspace of $L^2(X, m)$.
- $\mathcal{E} : \mathcal{F} \times \mathcal{F} \to \mathbb{R}$ is a non-negative definite, symmetric, and bilinear.
- $(\mathcal{E}, \mathcal{F})$ is closed ($\mathcal{F}$ is a Hilbert space under the inner product $\mathcal{E}_1 := \mathcal{E} + \langle \cdot, \cdot \rangle_{L^2(X,m)}$).
- $(\mathcal{E}, \mathcal{F})$ is Markovian:

$$f^+ \wedge 1 \in \mathcal{F} \text{ and } \mathcal{E}(f^+ \wedge 1, f^+ \wedge 1) \leq \mathcal{E}(f, f) \quad \text{for any } f \in \mathcal{F}.$$

# Dirichlet forms: regularity, strong-locality

- $(\mathcal{E}, \mathcal{F})$ is called regular if $\mathcal{F} \cap C_c(X)$ is dense both in $(\mathcal{F}, \mathcal{E}_1)$ and in $(C_c(X), \|\cdot\|_{\sup})$.

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- $(\mathcal{E}, \mathcal{F})$ is called strongly local if $\mathcal{E}(f, g) = 0$ for any $f, g \in \mathcal{F}$ with $\mathrm{supp}_m[f - a\mathbf{1}_X] \cap \mathrm{supp}_m[g] = \emptyset$ for some $a \in \mathbb{R}$.

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- MMD space is a metric measure space $(X, d, m)$ with a strongly local, regular, Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X, m)$.

# Fukushima's theorem '71

Every regular Dirichlet form has an associated symmetric Markov process.

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If the Dirichlet form is strongly local the Markov process has continuous sample paths (a diffusion process).

# Energy measure corresponding to a Dirichlet form

- The energy measure $\Gamma(f,f)$ of $f \in \mathcal{F} \cap L^\infty(X,m)$

  $$\int_X g\,d\Gamma(f,f) = \mathcal{E}(f,fg) - \frac{1}{2}\mathcal{E}(f^2,g) \quad \text{for all } g \in \mathcal{F} \cap C_c(X),$$

  and then by
  $\Gamma(f,f)(A) := \lim_{n\to\infty} \Gamma\big((-n)\vee(f\wedge n),(-n)\vee(f\wedge n)\big)(A)$
  for each Borel subset $A$ of $X$ for general $f \in \mathcal{F}$.
- For standard Brownian motion on $\mathbb{R}^n$

  $$\Gamma(f,f) = \frac{1}{2}\left|\nabla f\right|^2\,dm \ll m.$$

- Warning: $\Gamma(f,f) \perp m$ is possible.

# Local Dirichlet space on an open set $U$

$$\mathcal{F}_{\mathrm{loc}}(U) := \left\{ f \;\middle|\; \begin{array}{l} f \text{ is an } m\text{-equivalence class of functions on } U \text{ such} \\ \text{that } f\mathbb{1}_V = f^{\#}\mathbb{1}_V \ m\text{-a.e. for some } f^{\#} \in \mathcal{F} \text{ for} \\ \text{each relatively compact open subset } V \text{ of } U \end{array} \right\}$$

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The energy measure of a function $f \in \mathcal{F}_{\mathrm{loc}}(U)$ is defined as $\Gamma_U(f,f)(A) = \Gamma(f^{\#}, f^{\#})(A)$, for all $A \subset V$, with $V, f^{\#}$ as above.

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$$\mathcal{F}(U) := \{f \in \mathcal{F}_{\mathrm{loc}}(U) : \int_U f^2\, dm + \int_U \Gamma_U(f,f) < \infty\},$$

and the bilinear form $(\mathcal{E}_U, \mathcal{F}(U))$ as

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Warning: $(\mathcal{E}_U, \mathcal{F}(U))$ need not be a Dirichlet form on $L^2(\overline{U}, m|_{\overline{U}})$.

# Sub-Gaussian heat kernel estimates

We say that $(X, d, m, \mathcal{E}, \mathcal{F})$ satisfies the sub-Gaussian heat kernel estimates HKE($\beta$), if there exist $C_1, c_1, c_2, c_3, \delta \in (0, \infty)$ and a heat kernel $\{p_t\}_{t>0}$ such that for any $t > 0$, such that

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$$P_t f(x) = \int_X p_t(x,y) f(y)\, m(dy) \quad \text{for all } f \in L^2(X,m),$$

and

$$p_t(x,y) \le \frac{C_1}{m\big(B(x,t^{1/\beta})\big)} \exp\left(-c_1 \left(c_2 \frac{d(x,y)^\beta}{t}\right)^{1/(\beta-1)}\right)$$

$$p_t(x,y) \ge \frac{c_3}{m\big(B(x,t^{1/\beta})\big)} 1_{d(x,y) \le \delta t^{1/\beta}} \qquad \text{for } m\text{-a.e. } x, y \in X.$$

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Hino '05 showed that $\beta \geq 2$.

# Examples of sub-Gaussian heat kernel estimates

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- (Li-Yau '86) Riemannain manifolds with non-negative Ricci curvature satisfies Gaussian bounds HKE(2).
- (Barlow, Perkins '88) Brownian motion on the Sierpiński gasket satisfies HKE($\log_2 5$).
- (Barlow, Bass '99) Brownian motion on the Sierpiński carpet satisfies HKE($\beta$), where $\beta > 2$.
- Many other examples due to Barlow, Fitzsimmons, Hambly, Kumagai, Kigami, Lindstörm, ...

# Heat kernel estimate for reflected diffusion

Theorem (M'23+)

*Let* $(X, d, m, \mathcal{E}, \mathcal{F})$ *be a MMD space satisfying sub-Gaussian heat kernel estimate* HKE($\beta$), *where* $m$ *is a doubling measure and* $\beta \geq 2$.

1. *Then for any uniform domain* $U \subset X$, $(\mathcal{E}_U, \mathcal{F}(U))$ *is a strongly local regular Dirichlet form on* $L^2(\overline{U}, m)$.
2. *The MMD space* $(\overline{U}, d, m, \mathcal{E}_U, \mathcal{F}(U))$ *also satisfies sub-Gaussian heat kernel bounds* HKE($\beta$).

# Extension theorem

## Theorem (M. '23+)

*Let* $(X, d, m, \mathcal{E}, \mathcal{F})$ *be a MMD space that satisfies sub-Gaussian heat kernel bounds* HKE($\beta$), *where* $m$ *is a doubling measure. For any uniform domain* $U$, *there exists a bounded linear extension operator* $E : \mathcal{F}(U) \to \mathcal{F}$ *such that* $E(f)|_U = f$ *for all* $f \in \mathcal{F}(U)$.

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$$\Gamma(E(f), E(f))(B(x,r)) \le C\Gamma_U(f,f)(B_U(x,Kr)), \quad 0 < r < c\,\mathrm{diam}(U);$$

$$\int_{B(x,r)} |E(f)|^2 \, dm \le C \int_{B_U(x,Kr)} f^2 \, dm \quad \text{for all } r > 0;$$

$$\mathcal{E}(E(f), E(f)) \le C \left( \mathcal{E}_U(f,f) + \frac{1}{\mathrm{diam}(U)^\beta} \int_U f^2 \, dm \right);$$

$$\int_X |Ef|^2 \, dm \le C \int_X f^2 \, dm,$$

*where* $B_U(x, r) := U \cap B(x, r)$*.*

# Previous results

Theorem (Jones '81 Acta Math.)

*For any uniform domain* $U$ *in* $\mathbb{R}^n, k \in \mathbb{N}, p \in [1, \infty]$*, there exists a bounded linear extension map* $E : W^{k,p}(U) \to W^{k,p}(\mathbb{R}^n)$.

Extension theorem on Lipschitz domains is due to Calderón '69 and Stein '70. Similar extension result was obtained by Garofalo-Nhieu '98 for Carnot-Carathéodory spaces and by Björn-Shanmugalingam '07 for Sobolev space based on upper gradient $N^{1,p}$.

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Theorem (Gyrya, Saloff-Coste '11 Astérisque)

*Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space satisfying* HKE(2)*. Then for any uniform domain $U \subset X$, $(\mathcal{E}_U, \mathcal{F}(U))$ is a strongly local regular Dirichlet form on $L^2(\overline{U}, m)$. The MMD space $(\overline{U}, d, m|_{\overline{U}}, \mathcal{E}_U, \mathcal{F}(U))$ also satisfies Gaussian heat kernel bounds* HKE(2).

# A difference between $\beta = 2$ and $\beta > 2$

Theorem (Kajino, M.'20 Ann. Prob.)

*Let $(X, d, m)$ be a metric measure space with a m-symmetric diffusion that satisfies sub-Gaussian heat kernel bound* HKE($\beta$) *and such that d is bi-Lipschitz equivalent to a geodesic metric. Then*

1. *(Singularity) If $\beta > 2$, then $\Gamma(f, f) \perp m$ for all $f \in \mathcal{F}$.*
2. *(Absolute continuity) If $\beta = 2$, then $\Gamma(f, f) \ll m$ for all $f \in \mathcal{F}$.*

The singularity of energy measure was conjecture by M. Barlow '03.

# Remarks on the proof of extension theorem

- The construction of the extension operator is similar to the work of Jones using a partition of unity with 'low energy functions' and a quasi-conformal type reflection of Whitney covers.
- The proof of Jones and other earlier works rely on point-wise upper bounds on the gradient of the extended function to obtain upper bound on the Sobolev norm.
- Since the energy measure may be singular to the symmetric measure, we can not rely on point-wise bounds on gradient.
- How to estimate $\int_{\mathbb{R}^n} |\nabla f|^2 \, dm$ without estimating the distributional gradient $|\nabla f|$ for $f \in W^{1,2}(\mathbb{R}^n)$?
- Korevaar-Schoen theorem '93: For all $f \in W^{1,2}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} |\nabla f|^2 \, dm \asymp \liminf_{r \downarrow 0} \int_{\mathbb{R}^n} \fint_{B(x,r)} \frac{|f(x) - f(y)|^2}{r^2} \, m(dy) m(dx).$$

# Remarks on the proof of extension theorem

- **Theorem** (Grigor'yan, Hu and Lau '03) Let $(X, d, m, \mathcal{E}, \mathcal{F})$ satisfy the sub-Gaussian heat kernel estimate HKE($\beta$). A function $f \in L^2(X, m)$ belong to $\mathcal{F}$ if and only

$$\liminf_{r \downarrow 0} \int_X \fint_{B(x,r)} \frac{|f(x) - f(y)|^2}{r^\beta} m(dy) m(dx) < \infty.$$

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Furthermore, for all $f \in \mathcal{F}$

$$\mathcal{E}(f, f) \asymp \liminf_{r \downarrow 0} \int_X \fint_{B(x,r)} \frac{|f(x) - f(y)|^2}{r^\beta} m(dy) m(dx).$$

- Our proof that the extended function $E(f) \in \mathcal{F}$ relies on the above condition.
- We prove a version of the above estimate for energy measure.

# Proving heat kernel estimates using extension theorem

- Barlow, Bass, Kumagai '06, Grigor'yan, Hu, Lau '15: Given a MMD space $(X, d, m, \mathcal{E}, \mathcal{F})$ the sub-Gaussian heat kernel estimate HKE($\beta$) is equivalent to the doubling property of $m$, Poincaré inequality PI($\beta$) and cutoff energy inequality CS($\beta$).

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- **Message**: The extension theorem implies that functional inequalities are inherited by the domain from the ambient space.
- For an MMD space $(X, d, m, \mathcal{E}, \mathcal{F})$ the Poincaré inequality PI($\beta$) is as follows: there exist $C, A_1, A_2 \in (1, \infty)$ such that for all $x \in X, 0 < r < \mathrm{diam}(X, d)/A_2, f \in \mathcal{F}$, we have

$$\inf_{\alpha \in \mathbb{R}} \int_{B(x,r)} |f - \alpha|^2 \, dm \le C r^{\beta} \int_{B(x, A_1 r)} d\Gamma(f, f).$$

# Poincaré inequality for reflected diffusion

We will see how to prove Poincaré inequality for the MMD space $(\overline{U}, d, m|_{\overline{U}}, \mathcal{E}_U, \mathcal{F}(U))$ corresponding to the reflected diffusion using that for the ambient diffusion $(X, d, m, \mathcal{E}, \mathcal{F})$.

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$$\begin{aligned}
\inf_{\alpha\in\mathbb{R}} \int_{\overline{U}\cap B(x,r)} |f-\alpha|^2 \, dm &\le \inf_{\alpha\in\mathbb{R}} \int_{B(x,r)} |E(f)-\alpha|^2 \, dm \\
&\lesssim r^\beta \int_{B(x,A_1 r)} d\Gamma(E(f), E(f)) \\
&\lesssim r^\beta \int_{U\cap B(x,KA_1 r)} d\Gamma_U(f,f).
\end{aligned}$$

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$$\begin{aligned}
\inf_{\alpha \in \mathbb{R}} \int_{\overline{U} \cap B(x,r)} |f - \alpha|^2 \, dm &\leq \inf_{\alpha \in \mathbb{R}} \int_{B(x,r)} |E(f) - \alpha|^2 \, dm \\
&\lesssim r^\beta \int_{B(x,A_1 r)} d\Gamma(E(f), E(f)) \\
&\lesssim r^\beta \int_{U \cap B(x,KA_1 r)} d\Gamma_U(f, f).
\end{aligned}$$

# A conjecture of Grigor'yan, Hu, Lau '14

- It is also possible to prove the Poincaré inequality without the use of extension map using the approach of Gyrya and Saloff-Coste. However I do not know how to prove the cutoff energy inequality without using the extension theorem.

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- GHL conjecture: Given a MMD space $(X, d, m, \mathcal{E}, \mathcal{F})$ the sub-Gaussian heat kernel estimate HKE($\beta$) is equivalent to the doubling property of $m$, Poincaré inequality PI($\beta$) and the following capacity upper bound: there exists $C \in (0, \infty)$ such that

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- If the above conjecture were true, the proof of heat kernel bound for uniform domains can be simplified without relying on the extension theorem and can also be used to handle inner uniform domains.

# Thank you for your attention

M. Murugan, Heat kernel for reflected diffusion and extension property on uniform domains, arXiv:2304.03908.