Dimensions of Non-autonomous self-affine sets

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Compact $J \subset \mathbb{R}^d$ with $\text{int}(J) \neq \varnothing$. Integer sequence $\{n_k > 0\}_{k=1}^{\infty}$.

$\Sigma^k = \{u_1 u_2 \cdots u_k : 1 \leq u_j \leq n_j \ , j \leq k\}, \qquad \Sigma^* = \bigcup_{k=0}^{\infty} \Sigma^k$

$(c_{k,1}, c_{k,2}, \cdots, c_{k,n_k})$ satisfies $\sum_{j=1}^{n_k} (c_{k,j})^d \leq 1$ for each $k \in \mathbb{N}$.

The collection $\mathcal{F} = \{J_{\mathbf{u}} : \mathbf{u} \in \Sigma^*\}$ satisfies MSC:

(1). For each $\mathbf{u} \in \Sigma^*$, $J_{\mathbf{u}}$ is geometrically similar to J.

(2). For all $k \in \mathbb{N}$ and $\mathbf{u} \in \Sigma^{k-1}$, $J_{\mathbf{u}i} \subset J_{\mathbf{u}}$, $\text{int}(J_{\mathbf{u}i}) \cap \text{int}(J_{\mathbf{u}i'}) = \varnothing$ for $i \neq i'$, and, for all $1 \leq i \leq n_k$, $\frac{|J_{\mathbf{u}i}|}{|J_{\mathbf{u}}|} = c_{k,i}$,

The non-empty compact set

$$E = E(\mathcal{F}) = \bigcap_{k=1}^{\infty} \bigcup_{\mathbf{u} \in \Sigma^k} J_{\mathbf{u}}$$

is called a *Moran set* determined by $\mathcal{F}$.

Dimension formulas for Moran sets

For all $k' > k \geq 0$, let $d_{k,k'}$ be the unique real solution of the equation $\Delta_{k,k'}(d) = 1$, where

$$\Delta_{k,k'}(d) = \prod_{i=k+1}^{k'} \left(\sum_{j=1}^{n_i} (c_{i,j})^d \right). \tag{1}$$

Let d_*, d^* and d^{**} be the real numbers given respectively by

$$d_* = \liminf_{m\to\infty} d_{0,m}, \quad d^* = \limsup_{m\to\infty} d_{0,m}, \quad d^{**} = \lim_{m\to\infty} \left(\sup_k d_{k,k+m}\right).$$

Theorem (Feng, Hua, Li, Rao, Wen, Wu, Xi): *Given* $\inf\{c_{i,j}\} > 0$.

$$\dim_H E = d_*, \quad \dim_P E = \overline{\dim}_B E = d^*, \quad \dim_A E = d^{**}.$$

Self-affine sets

An iterated function system (IFS) is a finite family of contractions $S_1, \ldots, S_N : \mathbb{R}^d \to \mathbb{R}^d$ with $N \geq 2$. It is well-known that there is a non-empty compact set $E \subseteq \mathbb{R}^d$ such that

$$E = \bigcup_{i=1}^{N} S_i(E), \tag{2}$$

called the *attractor*.

If the IFS consists of affine contractions S_i: $\mathbb{R}^d \to \mathbb{R}^d$

$$S_i(x) = T_i(x) + a_i, \qquad i = 1, 2, \ldots, N, \tag{3}$$

where $a_i \in \mathbb{R}^d$ is a translation vector and T_i is a non-singular linear mapping, we call the attractor E *self-affine set*.

For $0 \leq s \leq d$, the *singular value function* of T is defined by

$$\phi^s(T) = \alpha_1(T)\alpha_2(T)\ldots\alpha_{m-1}(T)\alpha_m^{s-m+1}(T), \tag{4}$$

where m is the integer such that $m-1 < s \leq m$.

Let $d(T_1, \ldots, T_M)$ be the unique solution to

$$\lim_{k\to\infty}\left(\sum_{\mathbf{i}\in\mathcal{I}^k}\phi^s(T_{\mathbf{i}})\right)^{\frac{1}{k}} = 1,$$

which is often called *affine dimension* or *Falconer dimension*.

Theorem (Falconer, Solomyak): *Given* $\|T_i\| < \frac{1}{2}$ *for* $i = 1, 2, \dots M$, *it turns out that*

$$\dim_{\mathrm{H}} F = \dim_{\mathrm{B}} F = \min\{d, d(T_1, \dots, T_M)\}, \tag{5}$$

for almost all $\mathbf{a} = (a_1, \dots, a_M) \in \mathbb{R}^{Md}$.

Theorem (Jordan, Pollicott and Simon): *Given* $\|T_i\| < 1$ *for* $i = 1, 2, \dots M$, *it turns out that*

$$\dim_{\mathrm{H}} F(\omega) = \dim_{\mathrm{B}} F(\omega) = \min\{d, d(T_1, \dots, T_M)\}, \tag{6}$$

for almost all random perturbation ω.

Non-autonomous iterated function systems

Let $\{\Xi_k\}_{k\geq 1}$ be a sequence of collections of contractive mappings, that is

$$\Xi_k = \{S_{k,1}, S_{k,2}, \ldots, S_{k,n_k}\}, \tag{7}$$

where each $S_{k,j}$ satisfies that $|S_{k,j}(x) - S_{k,j}(y)| \leq c_{k,j}|x-y|$ for some $0 < c_{k,j} < 1$.

We say the collection $\mathcal{J} = \{J_{\mathbf{u}} : \mathbf{u} \in \Sigma^*\}$ of closed subsets of J fulfils the *Non-autonomous Moran structure* if it satisfies the following conditions:

(1). For all integers $k > 0$ and all $\mathbf{u} \in \Sigma^{k-1}$, the elements $J_{\mathbf{u}1}, J_{\mathbf{u}2}, \cdots, J_{\mathbf{u}n_k}$ of $\mathcal{J}$ are the subsets of $J_{\mathbf{u}}$. We write $J_{\emptyset} = J$ for the empty word $\emptyset$.

(2). For each $\mathbf{u} = u_1 \dots u_k \in \Sigma^*$, there exists an transformation $\Psi_{\mathbf{u}} : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$J_{\mathbf{u}} = \Psi_{\mathbf{u}}(J) = \Psi_{u_1} \circ \dots \circ \Psi_{u_j} \dots \circ \Psi_{u_k}(J),$$

where $\Psi_{u_j}(x) = S_{j,u_j} x + \omega_{u_1 \dots u_j}$, for some $\omega_{u_1 \dots u_j} \in \mathbb{R}^d$, and $S_{j,u_j} \in \Xi_j$, $j = 1, 2, \dots k$.

(3). The maximum of the diameters of $J_{\mathbf{u}}$ tends to 0 as $|\mathbf{u}|$ tends to ∞, that is,

$$\lim_{k \to \infty} \max_{\mathbf{u} \in \Sigma^k} |J_{\mathbf{u}}| = 0.$$

The non-empty compact set

$$E = E(\mathcal{J}) = \bigcap_{k=1}^{\infty} \bigcup_{\mathbf{u}\in\Sigma^k} J_{\mathbf{u}} \tag{8}$$

is called a *non-autonomous attractor* determined by $\mathcal{J}$. For all $\mathbf{u} \in \Sigma^k$, the elements $J_{\mathbf{u}}$ are called *kth-level basic sets* of E.

If the non-autonomous attractor E satisfies that for all $k \in \mathbb{N}$ and $\mathbf{u} \in \Sigma^{k-1}$,

$$\operatorname{int}(J_{\mathbf{u}i}) \cap \operatorname{int}(J_{\mathbf{u}i'}) = \emptyset \quad \textit{for } i \neq i' \in \{1, 2, \ldots, n_k\},$$

we say E satisfies *open set condition* (OSC).

Let $\{\Xi_k\}_{k\geq 1}$ be a sequence of collections of contractive matrices, that is

$$\Xi_k = \{T_{k,1}, T_{k,2}, \ldots, T_{k,n_k}\}, \tag{9}$$

where $T_{k,j}$ are $d \times d$ matrices with $\|T_{k,j}\| < 1$ for $j = 1, 2, \ldots n_k$.

We call the non-autonomous attractor E the *non- autonomous affine set or self-affine Moran set* determined by $\mathcal{J}$.

For all $\mathbf{u} \in \Sigma^k$, the elements $J_{\mathbf{u}}$ are called *kth-level basic sets* of E.

we always assume that

$$0 < \alpha_- \leq \alpha_+ < 1. \tag{10}$$

For each $s > 0$ and $0 < \epsilon < 1$, let m be the integer such that $m - 1 < s \leq m$ and define

$$\Sigma^*(s, \epsilon) = \{\mathbf{u} = u_1 \dots u_k \in \Sigma^* : \epsilon < \alpha_m(T_{\mathbf{u}^-}), \text{ and } \alpha_m(T_{\mathbf{u}}) \leq \epsilon\}.$$

We define

$$s^* = \inf \Big\{ s : \limsup_{\epsilon \to 0} \sum_{\mathbf{u} \in \Sigma^*(s,\epsilon)} \phi^s(T_{\mathbf{u}}) < \infty \Big\}. \tag{11}$$

Theorem: *Let E be the self-affine Moran set given by (8). Then*

$$\overline{\dim}_{\mathrm{B}} E \leq \min\{s^*, d\}.$$

Theorem: *Let E be the self-affine Moran set given by (8) and satisfying OPC. Suppose that there exists $c > 0$ such that*

$$\mathcal{L}^{d-1}\{\mathrm{proj}_{\Theta}(J \cap \Psi_{\mathbf{u}}^{-1}(E))\} \geq c,$$

for all $(d-1)$-dimensional subspaces Θ and all $\mathbf{u} \in \Sigma^*$. Then

$$\overline{\dim}_{\mathrm{B}} E = s^*.$$

For each integer $k > 0$, let

$$\mathcal{M}^s_{(k)}(G) = \inf\Big\{ \sum_{\mathbf{u}} \phi^s(T_{\mathbf{u}}) : G \subset \bigcup_{\mathbf{u}} \mathcal{C}_{\mathbf{u}}, |\mathbf{u}| \geq k \Big\}.$$

$$\mathcal{M}^s(G) = \lim_{k\to\infty} \mathcal{M}^s_{(k)}(G), \tag{12}$$

for all $G \subset \Sigma^\infty$.

We define

$$s_A = \inf\{s : \mathcal{M}^s(\Sigma^\infty) = 0\} = \sup\{s : \mathcal{M}^s(\Sigma^\infty) = \infty\}. \tag{13}$$

Hausdorff dimension

Theorem: *Let E be the self-affine Moran set given by (8). Then*

$$\dim_{\mathrm{H}} E \leq \min\{s_A, d\}.$$

Let $\mathcal{D}$ be a bounded region in $\mathbb{R}^d$. For each $\mathbf{u} \in \Sigma^*$, let $\omega_{\mathbf{u}} \in \mathcal{D}$ be a random vector distributed according to some Borel probability measure $P_{\mathbf{u}}$ that is absolutely continuous with respect to d-dimensional Lebesgue measure. We assume that the $\omega_{\mathbf{u}}$ are independent identically distributed random vectors. We let $\mathbf{P}$ denote the product probability measure $\mathbf{P} = \prod_{\mathbf{u} \in \Sigma^*} P_{\mathbf{u}}$ on the family $\omega = \{\omega_{\mathbf{u}} : \mathbf{u} \in \Sigma^*\}$. In this context, for each $\mathbf{u} = u_1 \dots u_k \in \Sigma^*$, we assume that the translation of Ψ_{u_j} is an element of ω, that is,

$$\Psi_{u_j}(x) = T_{j,u_j} x + \omega_{u_1 \dots u_j}, \qquad \omega_{u_1 \dots u_j} \in \omega = \{\omega_{\mathbf{u}} : \mathbf{u} \in \Sigma^*\},$$

for $j = 1, 2, \dots, k$.

Theorem: *Let E^{ω} be the self-affine Moran set with random translation. Then for $\mathbf{P}$-almost all ω,*

$$\dim_{\mathrm{H}} E^{\omega} = \min\{s_A, d\}$$

.

Comparison of critical values

Theorem: *The two critical values satisfy that*

$$s_A \leq s^*$$

where the inequality may hold strictly.

Theorem: *Suppose that* $\Xi_k = \{T_1, T_2, \ldots, T_M\}$ *for all* $k > 0$. *Then*

$$s_A = s^* = d(T_1, \ldots, T_M).$$

Thank you !