On Biaccessibility Dimension of Quadratic Julia Sets

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Dynamical Systems We Consider

- Unit Circle $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ in the complex plane $\mathbb{C}$.
- Doubling Map $\sigma_d(w) = w^d (d \geq 2)$; focusing on $d = 2$.
- Haar Measure on $\mathbb{S}^1 = \dfrac{\text{linear measure}(\cdot)}{2\pi}$; $\mathbf{h}(\sigma_d) = \log d$.
- Semi-conjugations $(\mathbb{S}^1, \sigma_d) \xrightarrow{\tau} (\mathcal{T}, g)$ such that $\mathcal{T}$ is a dendrite and $\{\tau^{-1}(u) : u \in \mathcal{T}\}$ a Good decomposition

Good means two properties:

(a) Convex Hulls of $\tau^{-1}(u_i)(u_1 \neq u_2)$ are disjoint.

(b) For all but one $u \in \mathcal{T}$ we have $\#g^{-1}(u) = d$ uncritical case

Let u_0 denote the only point with $\#g^{-1}(g(u_0)) = 1$ and $\mathcal{T}_0 \subset \mathcal{T}$ the smallest sub-continuum $\supset \{g^n(u_0) : n \geq 0\}$.

Then $g(\mathcal{T}_0) \subset \mathcal{T}_0$. Call $(\mathcal{T}_0, g)$ the dynamic core of $(\mathcal{T}, g)$.

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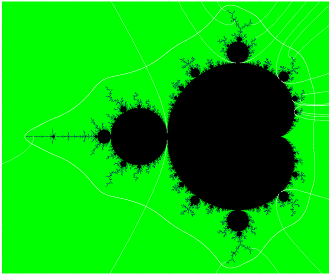
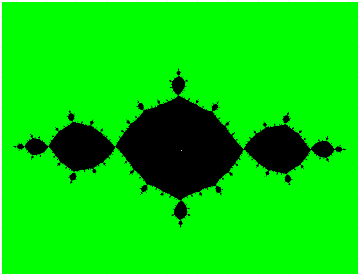
Identification of $(\mathbb{S}^1, \sigma_d) \xrightarrow{\mathcal{T}} (\mathcal{T}, g)$

Theorem (L-Tan-Yang-Yao(2023))

Given a polynomial f of degree $d \geq 2$. If the Julia set J is connected then (J, f) has a maximal dendrite factor, denoted by $\left(\mathcal{T}(f), \widetilde{f}\right)$.

Pick a parameter $c \in \mathcal{M}$, Mandelbrot set, we obtain $f_c(z) = z^2 + c$ and a semi-conjugation $(\mathbb{S}^1, \sigma_d) \xrightarrow{\mathcal{T} = \mathcal{T}_c} \left(\mathcal{T}(f_c), \widetilde{f_c}\right)$





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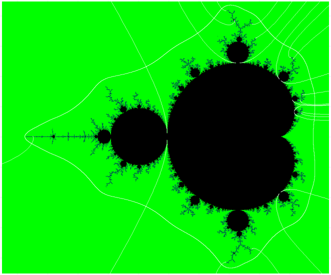

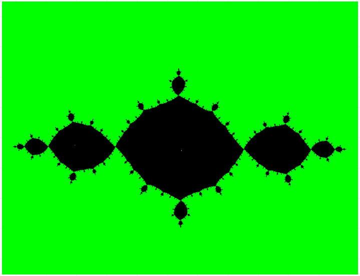

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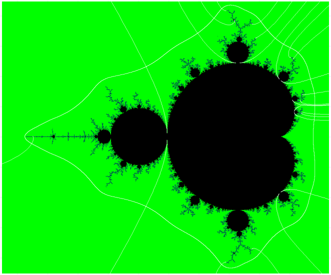

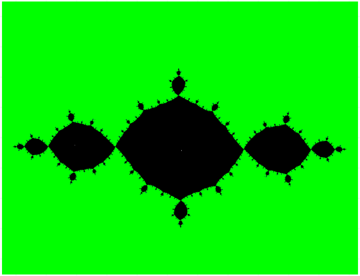

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Assume that 0 is the only critical point of f (unicritical). Let $u_0 = \pi(0)$ and $\mathcal{T}_0(f) \subset \mathcal{T}(f)$ the smallest sub-continuum containing $\mathbf{Orb}(u_0)$.

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How to define $(\mathbb{S}^1, \sigma_d) \xrightarrow{\tau} \left(\mathcal{T}(f), \widetilde{f}\right)$ by using $J \xrightarrow{\pi} \mathcal{T}(f)$ $\Leftarrow$ Explain

Note that $\xrightarrow{A(f)=\tau^{-1}(\mathcal{T}_0(f))} \sigma_d(A(f)) \subset A(f)$ and $\dim_H A(f) = \dfrac{ent\left(\sigma_d|_{A(f)}\right)}{\log d}$

See Proposition III.1 of [Furstenberg 1967]

Let $B(f) = \{\text{all } w \in A(f) \text{ with } \#\tau^{-1}(\tau(w)) > 1\}$.... If $\#\mathcal{T}(f) = 1$ details omitted

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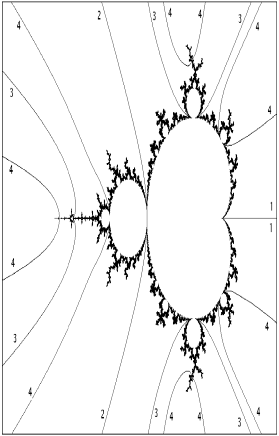

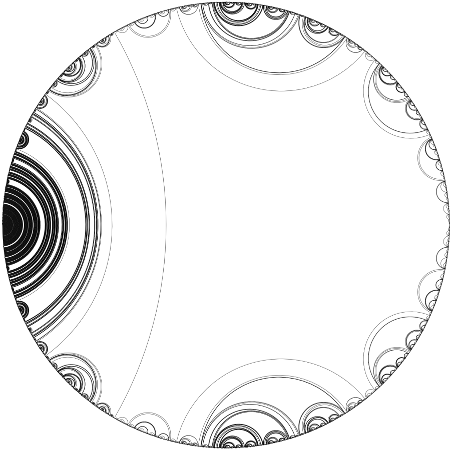

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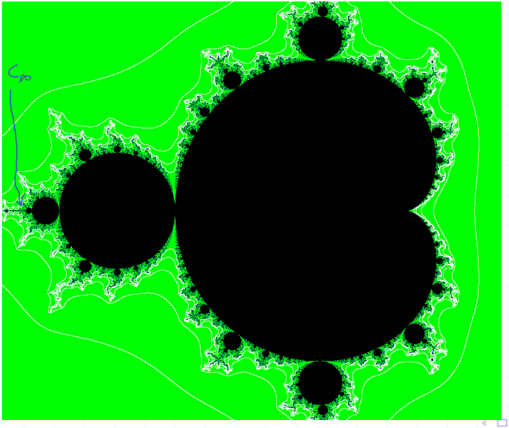

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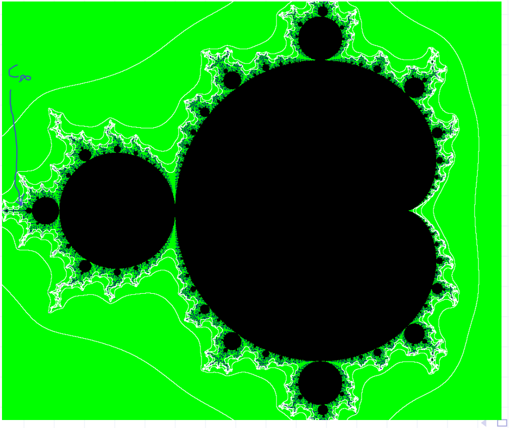

Central Molecule M_0

Complex Polynomials and Julia Sets

1. The filled Julia set K of a polynomial f, of degree $d \geq 2$, consists of all points $z \in \mathbb{C}$ such that **Orb**$(x) = \{f^n(z) : n \geq 0\}$ is bounded.

   $J = \partial K$ is the Julia set and $U_\infty = \mathbb{C} \setminus K$ the unbounded Fatou component.

2. Let $\sigma_d(w) = w^d$ and $\mathbb{D}^* = \{z : |z| > 1\}$. There is a unique conformal map ψ fixing ∞ and semi-conjugating $(\mathbb{D}^*, \sigma_d)$ with (U_∞, f).

3. f is PCF(post critically finite) if each critical point has a finite orbit.

4. If f is PCF then K and J are both connected and locally connected. Thus (J, f) is a factor of $(\mathbb{S}^1, \sigma_d)$, since ψ^{-1} may be continuously extended over $\mathbb{S}^1 = \partial\mathbb{D}^*$ by Caratheodory's Continuity Theorem.

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PCF Polynomials and Hubbard Trees

Theorem (Bielefeld-Fisher-Hubbard(1992))

If f is PCF then there is a tree $\mathcal{H}(f) \subset K$ (Hubbard tree) that is invariant under f and contains all the critical points of f.

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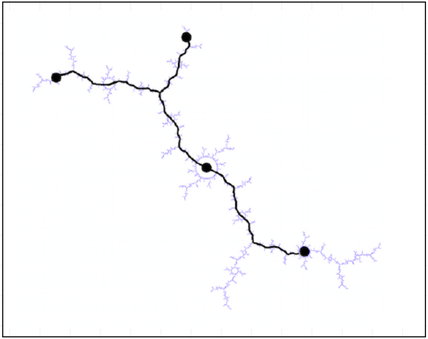
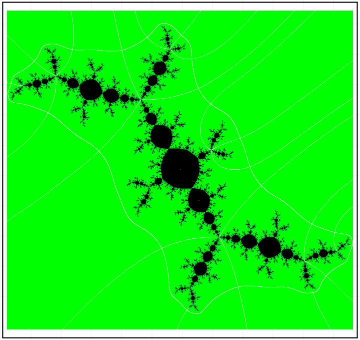
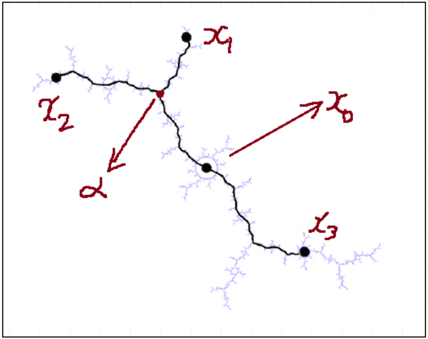
Example of Hubbard Tree $\mathcal{H}(f)$





- $f_c(z) = z^2 + c$ has an attractive 4-cycle; $f_c^4(0) = 0$; $\#\mathcal{T}_0(f_c) > 1$.
- $f_c(z) = z^2 + c$ has an attractive 6-cycle; $f_c^6(0) = 0$; $\#\mathcal{T}(f_c) = 1$.

Hubbard Tree & Incidence Matrix



0	0	0	1
1	0	0	1
1	1	0	0
0	0	1	0

- $x_4 = x_0, \quad f_c(x_i) = x_{i+1} (0 \leq i \leq 3)$.
- $a_1 = \overline{x_3 x_0}, \ a_2 = \overline{\alpha x_0}, \ a_3 = \overline{\alpha x_1}, \ a_4 = \overline{\alpha x_3}$.
- $(\mathcal{H}(f), f)$ and $\left(\mathcal{T}_0(f), \widetilde{f}\right)$ have the same entropy and the same matrix.

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- The wedge associated to θ, denoted by $\mathcal{W}_\theta$, consists of all the labelled pairs (i,j). Every pair (i,j) is called a vertex of $\mathcal{W}_\theta$.
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Theorem (Tiozzo(2016))

Thurston's entropy function $h_T : \mathbb{T} \cong \mathbb{S}^1 \to [0, \log 2]$ *is continuous.*

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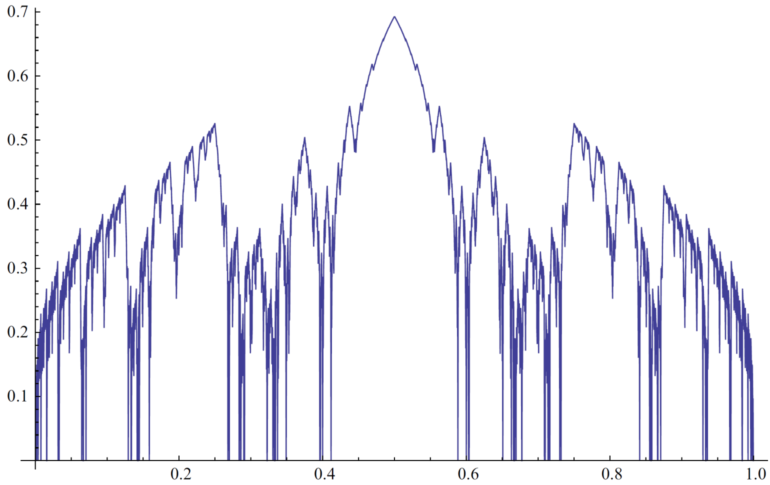
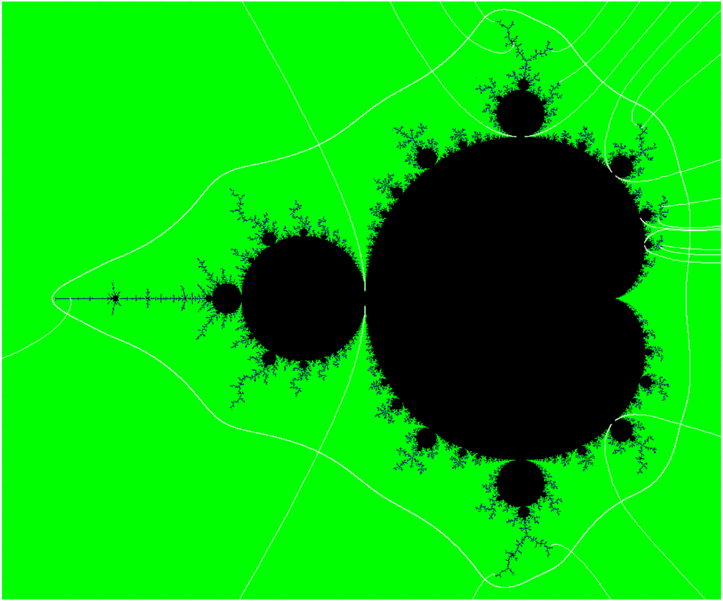
Thurston's Algorithm and Graph of h_T

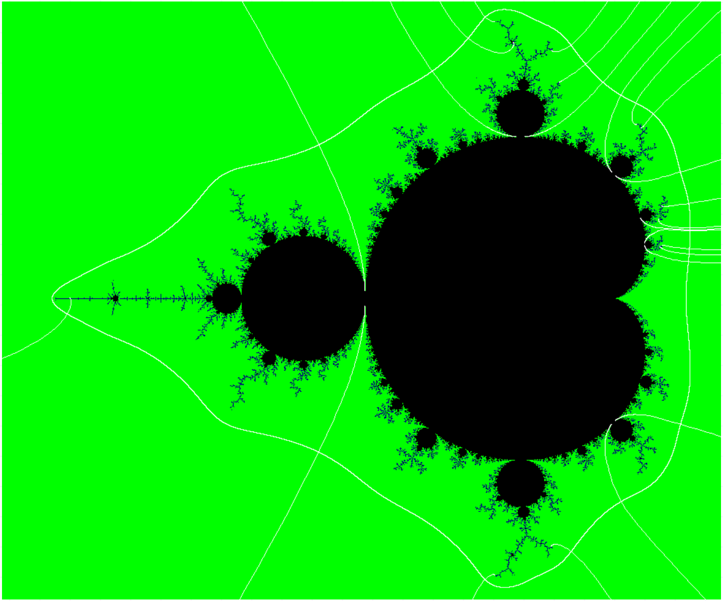


For $h_T(\theta)$ with $\theta \in \mathbb{Q}$. See Tiozzo(2016) and Gao(2020).

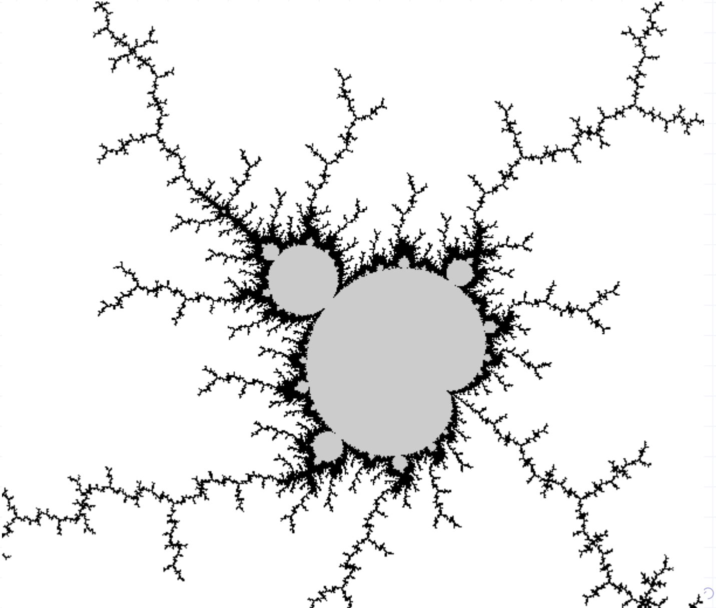
Structure of Mandelbrot Set $\mathcal{M}$



Entropy Map $h_T : \partial\mathcal{M} \to [0, \log 2]$



Extending h_T to the Whole $\mathcal{M}$



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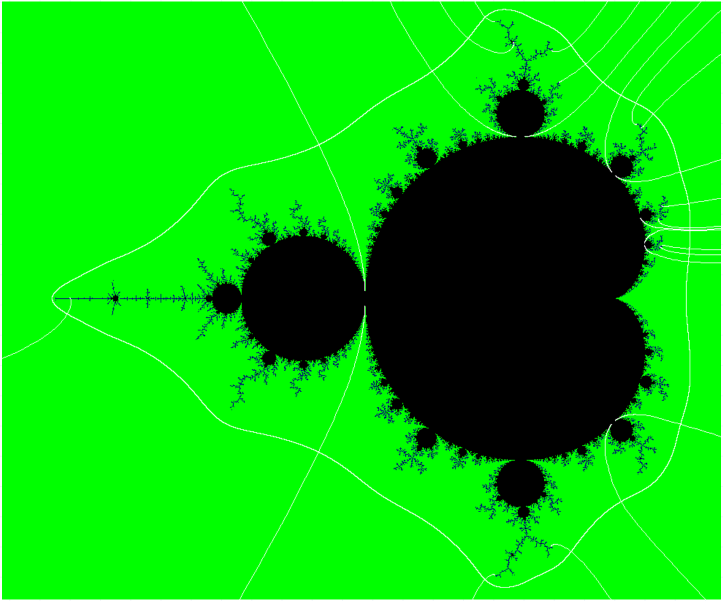

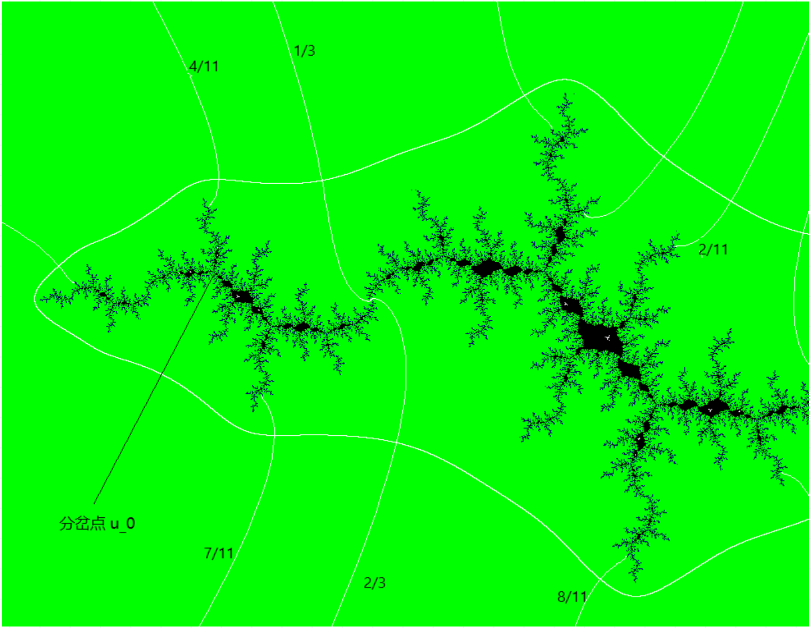
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谢 谢 大 家！

Thank You All