L^p-estimates of projections, Dual Furstenberg problem, and discretized sum-product

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Hausdorff dimension and Frostman measure

Denote by $\dim_{\mathcal{H}} E$ the Hausdorff dimension of an Euclidean subset E.

There are two equivalent ways to define $\dim_{\mathcal{H}} E$ through measures.

Frostman measure:

$$\dim_{\mathcal{H}} E = \sup\{s : \exists\, \mu \in \mathcal{M}(E),\ \sup_x \frac{\mu(B(x,r))}{r^s} < \infty\}.$$

Energy Integral:

$$\dim_{\mathcal{H}} E = \sup\{s : \exists\, \mu \in \mathcal{M}(E), I_s(\mu) < \infty\},\ \text{where } I_s(\mu) =$$

$$\iint |x-y|^{-s}\, d\mu(x)\, d\mu(y) = \int |\widehat{\mu}(\xi)|^2\, |\xi|^{-d+s}\, d\xi \left(= \|D^{-\frac{d-s}{2}}\mu\|_{L^2}^2 \right).$$

L^2-estimates of orthogonal projections

For every $e \in S^1$, denote by $\pi_e(x) = x \cdot e$ the orthogonal projection, and by $\pi_e\mu$ the pushforward of μ under π_e, i.e.,

$$\int f(t)\, d\pi_e\mu(t) = \int f(\pi_e(x))\, d\mu(x).$$

Marstrand (1954): Suppose $E \subset \mathbb{R}^2$ is Borel, $\dim_{\mathcal{H}} E > 1$, then for almost every $e \in S^1$, the set $\pi_e(E)$ has positive Lebesgue measure.

A classical argument of Kaufman (1968) states

$$\|\pi_e\mu\|^2_{L^2(\mathbb{R}\times S^1)} = c \iint |\hat{\mu}(te)|^2\, dt\, de = c \int |\hat{\mu}(\xi)|^2 |\xi|^{-1}\, d\xi = c \cdot I_1(\mu).$$

This tells more than the Marstrand projection theorem: $\pi_e\mu \in L^2(\mathbb{R})$ for almost every $e \in S^1$. Similar results hold in higher dimensions.

L^p-estimates of orthogonal projections

Dąbrowski, Orponen, Villa (2021): if μ is a Frostman measure on $\mathbb{R}^d$ of exponent $s > n$, then

$$\int \|\pi_V \mu\|_{L^p(\mathcal{H}^n)}^p \, d\gamma_{d,n}(V) < \infty, \ \forall\, 2 \leq p < 2 + \frac{s-n}{d-s}.$$

This range of p is in general optimal.

It has applications on incidence estimates (δ-discretized) and then on Furstenberg-type problems and sum-product problems.

The proof relies on Fourier multiplier theorem.

Later I gave an alternative proof with a more explicit upper bound.

s-dimensional amplitude

B.L., 2022: For $\mu \in \mathcal{M}(\mathbb{R}^d)$ and $s \in (0, d)$, define

$$A_s(\mu) := \sup_{x \in \mathbb{R}^d} \int |x - y|^{-s} \, d\mu(y) = \left(\|D^{-(d-s)}\mu\|_{L^\infty} \right).$$

I call $A_s(\mu)$ the s-dimensional amplitude of μ, or just s-amplitude.

This definition is very natural. It can be easily seen that

$$\dim_{\mathcal{H}} E = \sup\{s : \exists\, \mu \in \mathcal{M}(E),\ A_s(\mu) < \infty\}.$$

However there seems no previous discussion about this quantity.

B.L., 2022: Suppose μ is a measure and $p = 2 + \frac{s-n}{d-\alpha}$, then

$$\int \|\pi_V \mu\|^p_{L^p(\mathcal{H}^n)} \, d\gamma_{d,n}(V) \lesssim I_s(\mu) \cdot A_\alpha(\mu)^{p-2}.$$

A heuristic argument and analytic interpolation

Assuming $\mu \in C_0^\infty$, then $\pi_e\mu(t) = \int_{x\cdot e=t} \mu(x)\, d\mathcal{H}^1(x)$ and

$$\iint |\pi_e\mu(t)|^p \, dt\, de \le \left(\iint |\pi_e\mu(t)|^2 \, dt\, de\right) \cdot \|\pi_e\mu\|_{L^\infty}^{p-2}$$

$$= I_1(\mu) \cdot \|\pi_e\mu\|_{L^\infty}^{p-2} \le I_1(\mu) \cdot \|\mu\|_{L^\infty}^{p-2}.$$

By counting derivatives, it is approximately

$$I_1(D^{(p-2)(2-\alpha)/2}\mu) \cdot \|D^{-(2-\alpha)}\mu\|_{L^\infty}^{p-2} = I_{(p-2)(2-\alpha)+1}(\mu) \cdot A_\alpha(\mu)^{p-2},$$

which is $= I_s(\mu) \cdot A_\alpha(\mu)^{p-2}$ if $p = 2 + \frac{s-1}{2-\alpha}$.

Rigorously, apply analytic interpolation to $\Phi(z) = \int \pi_e\mu_z \cdot f_z$, with

$$\mu_z := \frac{\pi^{\frac{z}{2}}}{\Gamma(\frac{z}{2})} |\cdot|^{-d+z} * \mu(x) (= D^{-z}\mu)$$

the Riesz potential, between $f_z \in L^1$ and L^2.

Kakeya, Furstenberg and ite dual version

Recall the Kakeya/Besicovitch problem: a set containing a unit line segment in every direction must have full $\dim_{\mathcal{H}}$.

Furstenberg problem: Suppose $E \subset \mathbb{R}^2$ and there is a set of lines of dimension t such that $\dim_{\mathcal{H}}(l \cap E) \geq s$ for every line, then

$$\dim_{\mathcal{H}} E \geq \min(s + t, \frac{s + 3t}{2}, s + 1).$$

There are analogs in $\mathbb{R}^d$ intersecting k-planes. $k = d - 1$ is special.

We come up with its dual version: Suppose $\mathcal{V} \subset \mathbb{A}(d, k)$ is a set of k-planes in $\mathbb{R}^d$ and there is a set of points in $\mathbb{R}^d$ of dimension t such that $\dim_{\mathcal{H}}\{V \in \mathcal{V} : x \in V\} \geq s$ for every given x, then $\dim_{\mathcal{H}} \mathcal{V} \geq ?$.

$\mathbb{A}(d,k)$: the space of k-planes in $\mathbb{R}^d$

In the affine Grassmannian $\mathbb{A}(d,k)$, write $V = W + a$, with $W \in G(d,k)$, $a \in W^{\perp} \subset \mathbb{R}^d$, uniquely, then a natural measure is

$$\int_{A(d,k)} f = \int_{G(d,k)} \int_{W^{\perp}} f(W+a)\, d\mathcal{H}^{d-k}(a)\, d\gamma_{d,k}(W),$$

and a natural metric between $W + a$ and $W' + a'$ in $\mathbb{A}(d,k)$ is

$$d_{G(d+1,k+1)}\left(span\{(W,0),(a,1)\}, span\{(W',0),(a',1)\}\right)$$

Notice

$$\dim \mathbb{A}(d,k) = \dim G(d,k) + \dim \mathbb{R}^{d-k} = (k+1)(d-k),$$

which equals d if and only if $k = d-1$. In this case the Furstenberg problem and the dual version are equivalent (point-plane duality).

Incidence, Kakeya, and its dual

Suppose $\mathcal{P}$ is a union of δ-balls and $\mathcal{L}$ is a union of δ-tubes. What is

$$|I(\mathcal{P},\mathcal{L})| := |\{(p,l) \in \mathcal{P} \times \mathcal{L} : p \in \mathcal{N}_\delta(l)\}| \lesssim ?$$

Kakeya and incidence:

$$\delta^2 \lessapprox |I(\mathcal{P},\mathcal{L})| \lesssim \delta^2 \int_{\mathcal{P}} \sum \chi_T \leq \delta^2 \cdot |\mathcal{P}|^{\frac{1}{q'}} \cdot \| \sum \chi_T \|_{L^q}.$$

Then estimates of $\| \sum \chi_T \|_{L^q}$ imply dimension of Kakeya set.

For its dual, denote by $\mu_{\mathcal{P}} = \frac{1}{|\mathcal{P}|}\chi_{\mathcal{P}}$, we observe that

$$\delta^{(k+1)(d-k)-s} \lesssim |\mathcal{P}|^{-1} |\{(p,V) \in \mathcal{P} \times \mathcal{V} : p \in V\}|$$
$$= \int_{\mathcal{V} \subset \mathbb{A}(d,k)} \left(\int_V \mu_{\mathcal{P}} d\mathcal{H}^k \right) dV \leq |\mathcal{V}|^{\frac{1}{p'}} \cdot \|\pi_{W^\perp} \mu_{\mathcal{P}}\|_{L^p}.$$

Therefore L^p of orthogonal projections implies $\dim_{\mathcal{H}}$ in $\mathbb{A}(d,k)$.

Cartesian product

- The trivial estimate $\|\pi_e \mu\|_{L^\infty} \leq \|\mu\|_{L^\infty}$ can be improved when μ has special structure, e.g. $\mu = \mu_1 \times \mu_2$ of exponent $s = s_1 + s_2$.

The key observation is, if e is not vertical, then

$$\int_{x \cdot e = t} \mu_1(x_1)\mu_2(x_2)\, d\mathcal{H}^1(x) \leq \|\mu_1\|_{L^1} \cdot \|\mu_2\|_{L^\infty}.$$

Therefore when count derivatives, we only lose $1 - s_2$ derivatives, i.e.

$$\|\mu_1\|_{L^1} \cdot \|D^{1-s_2}\mu_2\|_{L^\infty} = \|\mu_1\|_{L^1} \cdot A_{s_2}(\mu_2),$$

while in general we have to lose $2 - s = (1 - s_1) + (1 - s_2)$ derivatives.

Sum-product and Elekes's approach via incidence

Suppose A is a finite subset of $\mathbb{R}$ (or $\mathbb{Z}$), it is conjectured that

$$\max\{\#(A+A), \#(A\cdot A)\} \gtrsim_\epsilon \#(A)^{2-\epsilon}.$$

Elekes (1997): apply incidence estimates (Szemerédi-Trotter) with

$$P = (A+A) \times (A \cdot A)$$

$$L = \{y = a_1(x - a_2) : a_1, a_2 \in A\} \sim A \times A.$$

This idea works on its continuous version, where A is a union of δ-intervals satisfying some non-concentration conditions.

We take advantages of the Cartesian product to obtain new results.

In the discrete setting, the structure of Cartesian product is not used. Also considering L^k estimates, $k \geq 2$ seems not to help.

The End