Plateau's Problem, calibrated sets, paired calibrated sets, and their product

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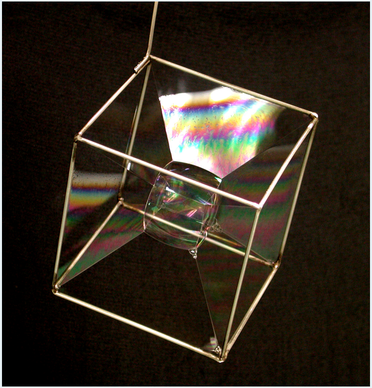
Classical geometric question

- The shortest object that connects two given points
- The shortest object that connects 3 or more given points ? (steiner)
- On general sets ? How to define "object" "shortest", "connects" ?
- On general dimension–the object that spans a circle with smallest area ; what about others ?
- How to define "object" "area" "span"

Back ground—Plateau's problem

Aim : try to understand regularity and existence of physical objects that have certain minimizing properties, such as soap films (minimizing "surface area" while spanning a given boundary).

Soap films

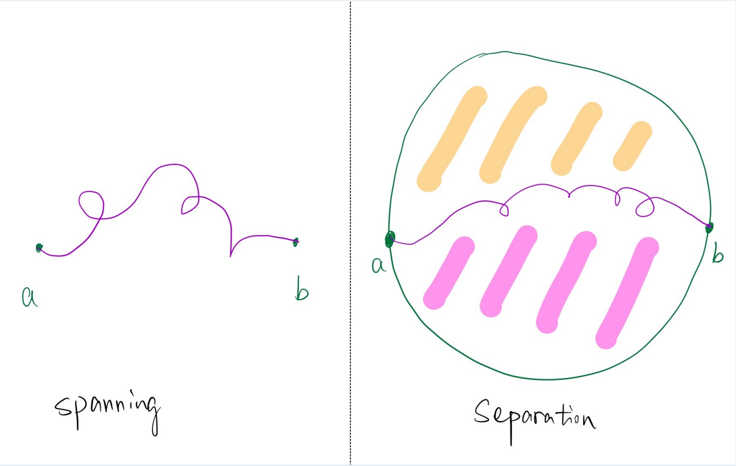


Mathematical intepretations–spanning

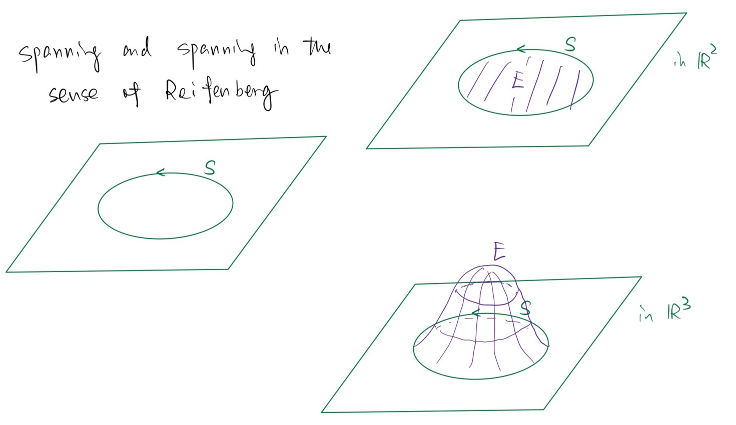
Take 1-dimensional objects for an simple idea :

- Curves, curve length, endpoints as boundary : the class is too small, not for 3 points, etc.
- other concepts to express "connect" or "spanning" : topological intepretation ? Given a set Γ, denote by its boundary $\bar{\Gamma}\setminus\Gamma$? Again the class is not closed
- Reifenberg's intepretation (sets, homological spanning, Hausdorff measure)
- Currents : via Stokes formula (or integration by parts) (De Rham, Federer, Fleming)
- Separation (Mumford-Shah)

connecting and separation



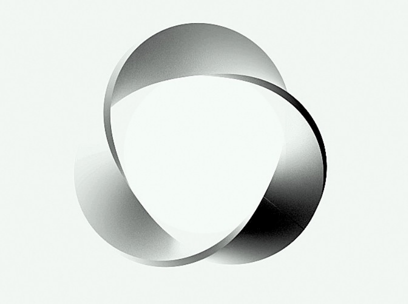
Spanning, and in the sense of Reifenberg : filling "holes"

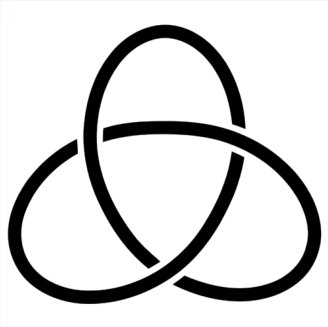


Remark

Both "spanning" and "separation" ask orientation or even groups. But why should soap films know about these ?

Non-orientable soap film





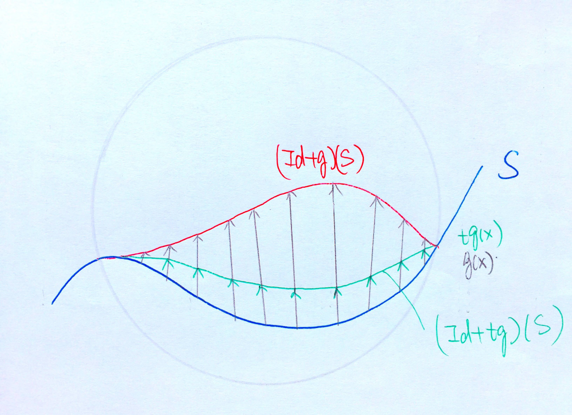
Back to deformations

- First variation of measure along deformations : minimal surface (Douglas, Fields Medal 1936), stationary varifold (Almgren, Allard)
- Minimal sets : absolute minima among deformations (Almgren)

Back to deformation-minimal surfaces

Let S be an k dimensional surface in $\mathbb{R}^n$, the first variation δS for the area functional on S is a linear form on $C_C(\mathbb{R}^n, \mathbb{R}^n)$:

$$\begin{aligned} \delta S(g) &= \frac{d}{dt}\Big|_{t=0} Area(S_t) \\ &= \frac{d}{dt}\Big|_{t=0} \int_S |D(Id + tg)(x)(TxS)|dx, \forall g \in C_C^\infty(\mathbb{R}^n, \mathbb{R}^n). \end{aligned}$$



Back to deformation : (Almgren-)minimal sets

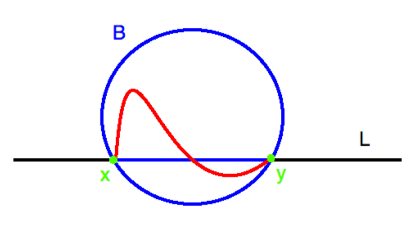
A closed set in a domain $U \subset \mathbb{R}^n$ is said to be minimal of dimension d in U if its d-dimensional Hausdorff measure cannot be decreased by any Lipschitz deformation with compact support.

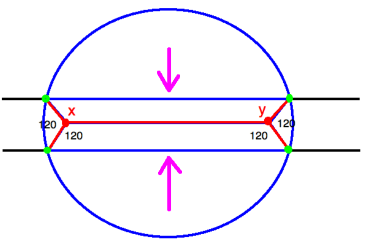
Definition

A closed set $E \subset \mathbb{R}^n$ is said to be a d-dimensional (Almgren-)minimal set in $\mathbb{R}^n$ if for every compact ball B, and every Lipschitz deformation f in B,
(i.e. $f : \mathbb{R}^n \to \mathbb{R}^n$, $f|_{B^C} = id$ and $f(B) \subset B$)

$$H^d(E \cap B) \leq H^d(f(E) \cap B).$$

Example (Lipschitz deformations in a ball)





Regularity, tangent cones

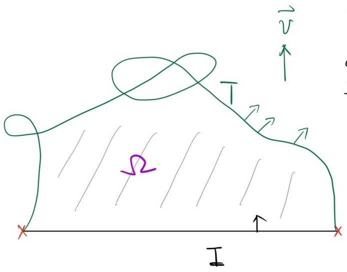
Main theory of local structure

- Rectifiability, existence of tangent planes a.e., monotonicity of density ratio $r \mapsto \frac{\mathcal{H}^k(E \cap B(x,r))}{r^k}$, and thus existence of density on EVERY point x Almgren, David-Semmes
- Existence of "tangent cone" (Hausdorff limits of $\frac{1}{r}(E - x)$ as $r \to 0$), classification of singularities Almgren, David-Semmes
- Open problem : uniqueness of tangent cone Allard, Almgren, Preiss, Simon, White, etc.
- Stratification on singularity : the dimension, d-rectifiability, $C^{1,\alpha}$-regularity for the singular set of points whose tangent cones are all of the form $C \times \mathbb{R}^d$ Naber & Valtorta
- General regularity and local parametrization David, Hardt, Simon and many others

How to prove minimality

- For stationary points : verify a pde, but not minima
- For minimal ones, it is in general hard
- Calibrations : for spanning Harvey & Lawson 1981, Calibrated geometry ; Lawlor 1991, Calibration and area decreasing flow ; Hardt & Simon : Foliation for minimal surfaces
- Paired calibrations : for separation Brakke 1991, Lawlor & Morgan 1994
- And : soap films is oriented or two sided

Examples



$\partial(T-I)=0$

$\Rightarrow \exists \Omega$ s.t. $\partial\Omega = T - I$

$\xrightarrow{\text{Stokes}} \int_{T-I} \vec{v} = \int_\Omega d\vec{v} = 0$

$\Rightarrow \int_I \vec{v} = \int_T \vec{v}$

在T上每个点有 $\langle \vec{v}, \vec{T} \rangle \leq \|\vec{v}\| \|\vec{T}\| = 1$

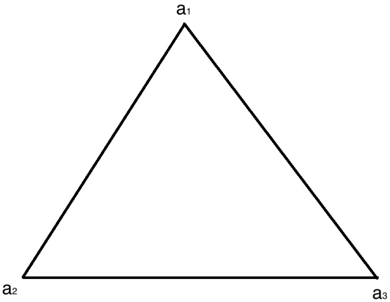
而在I上每个点均有 $\langle \vec{v}, \vec{I} \rangle = \|\vec{v}\| \|\vec{I}\| = 1$ calibration

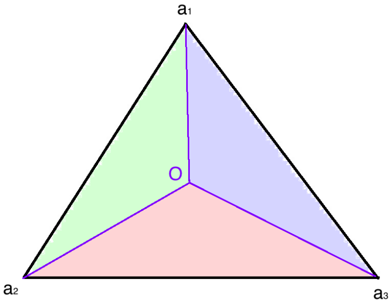
$\Rightarrow \mathcal{H}^1(I) = \int_I \langle \vec{I}, \vec{v} \rangle = \int_I \vec{v} = \int_T \vec{v} = \int_T \langle \vec{T}, \vec{v} \rangle \leq \int_T 1 = \mathcal{H}^1(T)$

Examples

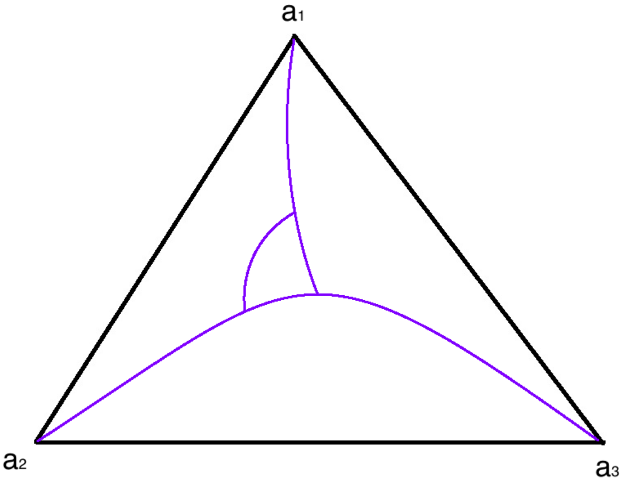
- Dimension 1 : a vector field whose integral depends only on endpoints but not on paths ;
- Higher dimension : closed forms : a disk ;
- Minimal graph
- $P_1 \cup_\perp P_2$ in $\mathbb{R}^4$
- Calibration with singularitiy : the simons cone : the cone over $S^3 \times S^3$ in $\mathbb{R}^8$: $C = \{x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_5^2 + x_6^2 + x_7^2 + x_8^2\}$
- All calibrations for cones are with singularities

Take the 1-dimensional $\mathbb{Y}$ set for example. So take a unit equilateral triangle C centered at the origin with vertices $a_i, 1 \leq i \leq 3$, with $|a_i| = 1, 1 \leq i \leq 3$. We want to prove that the set $Y = \cup_{i=1}^{3}[o, a_i]$ is minimal among all the (1-dimensional) sets $E \subset C$ that separate the three edges $[a_1, a_2], [a_2, a_3]$ and $[a_3, a_1]$. That is, the three edges belong to 3 different connected components of $C \backslash E$.



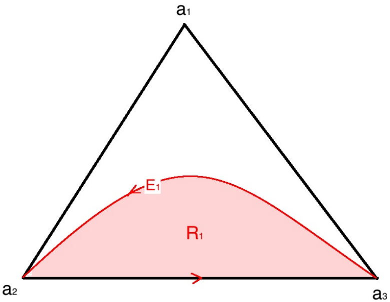


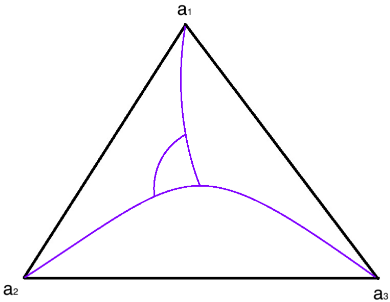
Take a set $E \subset C$ that separates the three edges.



Take a connected component R_1 of $C \backslash E$ that contains $[a_2, a_3]$. Let E_1 be the curve (contained in E) such that

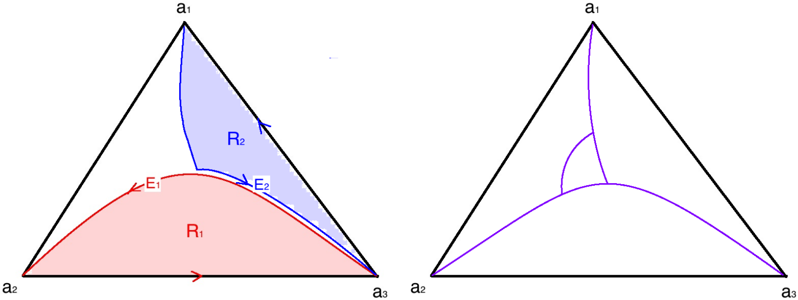
$$\partial R_1 = E_1 \cup [a_2, a_3].$$





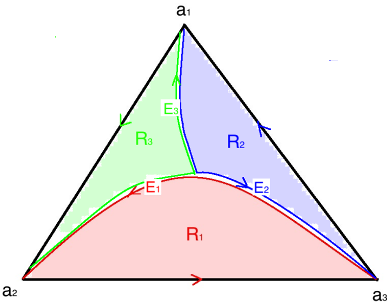
The same : take a connected component R_2 of $C \setminus E$ that contains $[a_3, a_1]$. Let E_2 be the curve (contained in E) such that

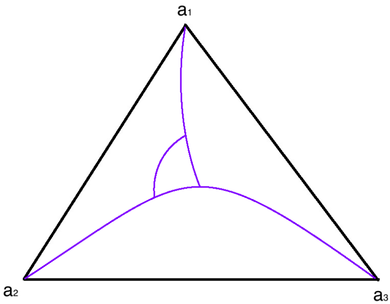
$$\partial R_2 = E_2 \cup [a_3, a_1].$$



Last, let $R_3 = C \backslash (R_1 \cup R_2 \cup E)$. Let E_3 be the curve (contained in E) such that

$$\partial R_3 = E_3 \cup [a_1, a_2]$$



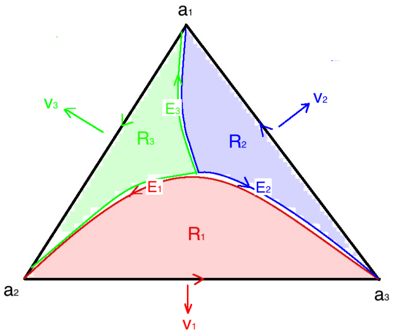


Now let $v_i = \overrightarrow{a_i o}, 1 = 1, 2, 3$. Then by divergence theorem

$$0 = \int_{\partial R_1} < \vec{n}_1, v_1 > = \int_{[a_2, a_3]} < \vec{n}_1, v_1 > + \int_{E_1} < \vec{n}_1, v_1 >,$$

where $\vec{n}_1$ is the unit outer normal vector of R_1.
Hence

$$\int_{E_1} < -\vec{n}_1, v_1 > = \int_{[a_2, a_3]} < \vec{n}_1, v_1 > = \sqrt{3}.$$



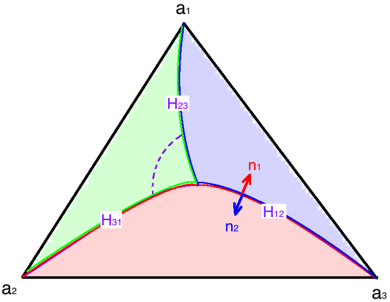
By the same reason,
$\int_{E_2} < -\vec{n}_2, v_2 >= \sqrt{3} = \int_{E_3} < -\vec{n}_3, v_3 >$. Hence

$$\sum_{i=1}^{3} \int_{E_i} < -\vec{n}_i, v_i >= 3\sqrt{3}. \tag{1}$$

Now for $1 \leq i \neq j \leq 3$, let $H_{ij} = E_i \cap E_j$. Note that the sets H_{ij} are essentially disjoint,

$$E_1 = H_{12} \cup H_{13}, E_2 = H_{23} \cup H_{12}, E_3 = H_{13} \cup H_{23},$$

and on H_{ij}, $\vec{n}_i = -\vec{n}_j$.



How to prove minimality–paired calibration for separation

Therefore (1) becomes

$$(2)\ \int_{H_{12}} < \vec{n}_1, v_2 - v_1 > + \int_{H_{23}} < \vec{n}_2, v_3 - v_2 > + \int_{H_{31}} < \vec{n}_3, v_1 - v_3 >= 3\sqrt{3}.$$

For $i \neq j$, $|v_i - v_j| = \sqrt{3}$, hence $| < \vec{n}_i, v_j - v_1 > | \leq |\vec{n}_i| \cdot |v_j - v_i| = \sqrt{3}$.
Therefore

$$H^1(H_{ij}) = \frac{1}{\sqrt{3}} \int_{H_{ij}} \sqrt{3} \geq \frac{1}{\sqrt{3}} \int_{H_{ij}} < \vec{n}_i, v_j - v_1 >,$$

with equality only if n_j is parallel to $v_j - v_i$ (which is the case for the $\mathbb{Y}$ set).

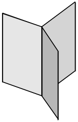
$$\begin{aligned} H^1(E) &\geq H^1(H_{12}) + H^1(H_{23}) + H^1(H_{31}) \\ &\geq \frac{1}{\sqrt{3}}\left(\int_{H_{12}} < \vec{n}_1, v_2 - v_1 > + \int_{H_{23}} < \vec{n}_2, v_3 - v_2 > + \int_{H_{31}} < \vec{n}_3, v_1 - v_3 >\right) \\ &= 3 = H^1(Y). \end{aligned}$$

Disadvantage : the product of paired calibrations do not form another set of paired calibrations any more. See below.

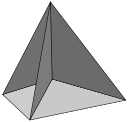
Minimal cones of codimension 1–paired calibration for separation

- 2-dimensional minimal cones in $\mathbb{R}^3$: the list is known for over a century

a plane

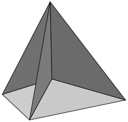


$\mathbb{Y}$



$\mathbb{T}$

- Higher dimensions : Cone over the $n-2$ dimensional skeleton of a regular simplex in $\mathbb{R}^n$ (G.Lawlor & F. Morgan 1994), cone over the $n-2$ dimensional skeleton of a cube in $\mathbb{R}^n$ for $n \geq 4$ (K. Brakke 1991)



simplex

cube

competitor

How to find new singularities–taking product

Essential difficulty : in codim 2, there is no separation, and no sides for faces, hence the multiplicity≤ 2 for paired calibrations no longer holds.

Product

- Product of calibrated sets :
- Product of paired calibrated sets : codim 2, and no separation, it is hard. Up to now we only know the minimality for $\mathbb{Y} \times \mathbb{Y}$ (L. 2014)
- Product of a codim 1 calibrated manifold (no singularity) with a paired calibrated set : spanning meets separation. Use projection along integral curves, to get a separation. But does not work for calibrated minimal cones because cones are with singularities (L.2022)
- Product of a codim 1 calibrated set (with singularity) with a paired calibrated set—-to obtain new tangent cones. (L. 2023)
- Product of two paired calibrated sets (with singularity) (L.2023)

Open questions

- Product of general minimal sets ?
- Does there exist a "calibration" for every "spanning" minimal set ?
- Does there exist "paired calibrations" for every "separation" minimal set ?
- And the product ?

Thank you !