On Erdő's similarity problem and its variants

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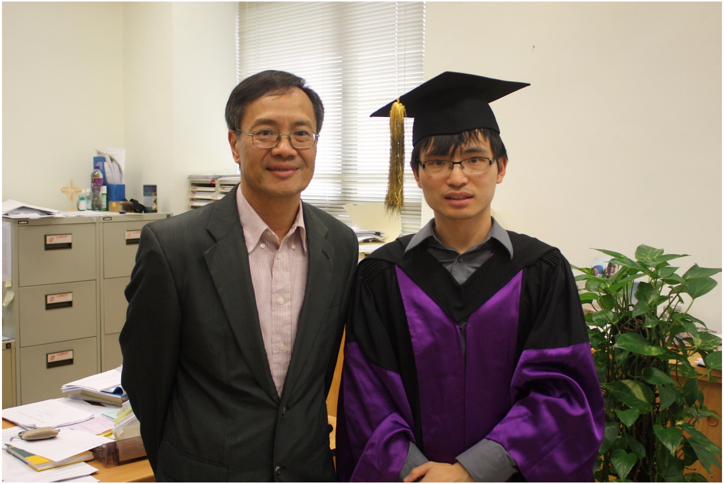

Dedictated to the Memory of Professor Ka-Sing Lau

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E.g. A bounded set is universal in $\mathcal{X} = \{\text{set with non-empty interior}\}$

Measure universality

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Theorem (Steinhaus, 1920)

Let K be a measurable set of positive Lebesgue measure on $\mathbb{R}^1$, and P be any finite sets. Then K contains an affine copy of P.

Indeed, the set

$$\{x \in K : \exists \delta \neq 0 \ s.t. \ x + \delta P \subset E\}.$$

contains all Lebesgue points and has full Lebesgue measure.

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He bet $100 for a proof.

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5. It is still open for $\{2^{-n} : n = 1, 2, ...\}$.

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4. Bradford, Kohut and Mooroogen considered the problem "in the large". They showed that for any unbounded sequence of certain restricted increasing rate and $p \in (0, 1)$, there always exists $\mathcal{L}(E) > 0$ such that $\mathcal{L}(E \cap [x, x+1]) \geq p$ for all $x \in \mathbb{R}$ and E does not contain an affine copy of this sequence.

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5. Improving this result, Keleti et al showed there exists a Lebesgue point in a set E such that there is no affine copy of $\{2^{-n}\}$ around that point.

Related problems

1. (Keleti, 1999) There exists a compact set of H-dim 1 not containing arithmetic progression of 3-terms.
2. (Keleti, 2009) There exists a compact set of H-dim 1 not containing any affine copy of a given countable pattern.

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6. (Pramanik, Liang, 2022) A rationally independent compact set must have zero Fourier dimension.

Main Result

Bi-Lipschitz Embedding

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Let's consider **bi-Lipschitz maps**: $\exists L > 1$ such that

$$L^{-1}|x-y| \le |f(x)-f(y)| \le L|x-y| \quad \text{for all } \ x, y \in \mathbb{R}.$$

Definition

We say that A is **bi-Lipschitz measure universal** *if A can be bi-Lipschitz embedded into every measurable set of positive Lebesgue measures. i.e. for all $\mathcal{L}(E) > 0$, there exists bi-Lipschitz f such that $f(A) \subset E$.*

1. Fast decaying sequences: Yes

Theorem (Feng, L. Xiong, 2023)

Let $(a_n)_{n=1}^{\infty}$ be a strictly decreasing sequence of positive numbers with $a_n \to 0$ as $n \to \infty$. If there exists an integer $N \geq 1$ such that

$$\limsup_{n\to\infty} \frac{a_{n+N}}{a_n} < 1,$$

then for any measurable set $E \subset \mathbb{R}$ with positive Lebesgue measure, there exists a bi-Lipschitz map $f : \mathbb{R} \to \mathbb{R}$ such that $f(a_n) \in E$ for all $n \geq 1$ and $f'(0) = 1$.

2. Slow Decaying sequence: No!

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then there exists a compact set $E \subset \mathbb{R}$ with positive Lebesgue measure such that $(a_n)_{n=1}^{\infty}$ can not be bi-Lipschitz embedded into E.

Sketch of Proof (Fast sequence):

We only work on $N = 1$. Assumption tells us that there exists $\delta < 1$ such that

$$\frac{a_{n+1}}{a_n} < \delta < 1.$$

i.e. $[\delta a_{n+1}, a_{n+1}] \cap [\delta a_n, a_n] = \emptyset$. Let $I_n = [\delta a_n, a_n]$.

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Without loss of generality, assume 0 is the Lebesgue point. We have

$$\lim_{n\to\infty} \frac{\mathcal{L}(E \cap I_n)}{\mathcal{L}(I_n)} = 1, \text{ or } \lim_{n\to\infty} \frac{\mathcal{L}(I_n \setminus E)}{\mathcal{L}(I_n)} = 0.$$

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If we pick a point $b_n \in I_n$ and define $f(a_n) = b_n$, we obtain a bi-Lipschitz map by linear interpolation. If we want to make sure $f'(0) = 1$, we pick b_n much closer to a_n as $n \to \infty$, which is possible by the density condition.

Sketch of Proof (Slow sequence):

Proposition

Let $(a_n)_{n=1}^{\infty}$ be a strictly decreasing sequence of positive numbers with $a_n \to 0$ as $n \to \infty$. Suppose that

$$\sup_{m>n>1} \frac{a_{m-1} - a_m}{a_{n-1} - a_n} < \infty, \quad \text{and} \quad \limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = 1.$$

Then there exists a compact set $E \subset \mathbb{R}$ with positive Lebesgue measure such that $(a_n)_{n=1}^{\infty}$ can not be bi-Lipschitz embedded into E.

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$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1 \implies \exists a_{n_k}$ such that

$$\lim_{k\to\infty} \frac{a_{n_k+1}}{a_{n_k}} = 1, \text{ and } a_{n_k} - a_{n_{k+1}} \le 2a_{n_m} - a_{n_{m+1}} \ \forall k > m.$$

Sketch of Proof (Slow sequence):

Assumption implies

$$a_n - a_{n+1} \leq C(a_m - a_{m+1}) \quad \text{for all } n > m.$$

$$\liminf_{n\to\infty} \frac{a_n - a_{n+1}}{a_n} = 0,$$

Choose a subsequence $(n_k)_{k=1}^{\infty}$ such that

$$\frac{a_{n_k} - a_{n_k+1}}{a_{n_k}} \leq k^{-2}4^{-k} \quad \text{for } \ k \geq 1.$$

Let

$$\ell_k = \lceil k/a_{n_k} \rceil, \ \delta_k = k(a_{n_k} - a_{n_k+1})$$

Then

$$\frac{1}{\ell_k} \leq \frac{a_{n_k}}{k} < \frac{2}{\ell_k}, \ \ell_k\delta_k < 2 \cdot 4^{-k}.$$

Sketch of Proof (Slow sequence):

Define $E = \bigcap_{k=1}^{\infty} E_k$ where

$$E_k = [0,1] \setminus \bigcup_{j=0}^{\ell_k} \left(\frac{j}{\ell_k} - \frac{\delta_k}{2}, \frac{j}{\ell_k} + \frac{\delta_k}{2} \right).$$

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Suppose we can biLipschitz embed (a_n) into E (thus also E_k).

1. $\mathcal{L}(E) \geq 1 - \sum_{k=1}^{\infty} \mathcal{L}\left([0,1] \setminus E_k]\right) > 0.$

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1. $\mathcal{L}(E) \geq 1 - \sum_{k=1}^{\infty} \mathcal{L}\left([0,1] \setminus E_k]\right) > 0.$
2. Upper Lipschitz bound: for all $m > n_k$ and $k > CL$

$$|b_m - b_{m+1}| \leq L|a_m - a_{m+1}| \leq CL|a_{n_k} - a_{n_{k+1}}| \leq k(a_{n_k} - a_{n_{k+1}}) = \delta_k.$$

(b_m cannot jump the gap, so all b_m all in one basic interval at stage k, so is its limit)

Sketch of Proof (Slow sequence):

Lower Lipschitz bound:

$$|b_{n_k} - b_\infty| \geq \frac{a_{n_k}}{L} \geq \frac{a_{n_k}}{k} > \frac{1}{\ell_k}$$

It must jump over some gaps of E_k, contradiction.

Infinite limit points

We also find a set with infinitely many limit points that are bi-Lipschitz measure universal.

Theorem (Feng, L. and Xiong, 2023)

Let $A = (a_n)_{n=1}^{\infty}$ *be a sequence of positive numbers such that*

$$a_1 + \sum_{n=1}^{\infty} \frac{a_{n+1}}{a_n} < \frac{1}{8}.$$

Then the set

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For the sake of time, we are not going to prove this theorem.

Main Result

Main Results II: Cantor Sets

Cantor sets

Theorem (Gallagher, L. and Weber, 2022)

There exists a dense G_δ set G with $\mathcal{L}(\mathbb{R} \setminus G) = 0$ such that it does not contain any Cantor sets of positive Newhouse thickness.

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Definition (Newhouse thickness)

Let u be an endpoint of a bounded complement interval G of K. The bridge B is the interval until you hit the interval whose length is at least $|G|$

$$\tau(K,u) = \frac{|B|}{|G|}, \ \tau(K) = \inf \tau(K,u).$$

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Corollary

If a Cantor set is bi-Lipschitz measure universal, then it must have Newhouse thickness zero.

Cantor sets

Lemma (Newhouse Gap Lemma)

Let K_1, K_2 be two Cantor sets such that $\tau(K_1)\tau(K_2) \geq 1$. Suppose that one is not contained in the gap of the other. Then $K_1 \cap K_2 \neq \emptyset$.

Sketch of Proof of GLW Theorem:

1. Let K_N be Cantor set with $\tau(K_n) \geq N$, $\text{conv}(K_n) = [0, 1]$ and $\mathcal{L}(E) = 0$.

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4. $\mathcal{L}(F) = 0$, the complement is the desired set.

Main Result

Main Results III: Topological universality

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4. Sets with interior is not topologically universal.

Topological Erdős universality problem

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Idea of the Proof:

1. For every gauge function g, there exists a dense G_δ set E such that $\mathcal{H}^g(E) = 0$.
2. For every Cantor set K, there always exists a gauge function such that $\mathcal{H}^g(K) > 0$.
3. Need Baire Category theorem to take care of all affine transformations.

Topological Erdős universality problem

This is a classical definition traced back to Borel.

Definition

A subset X of $\mathbb{R}$ is a strong measure zero set if for each sequence $(\epsilon_n)_n$ of positive reals, there exists a sequence of intervals $(I_n)_n$ such that $X \subset \cup_{n\in\mathbb{N}} I_n$ and $|I_n| < \epsilon_n$.

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Theorem (Galvin-Mycielski-Solovay, 1973)

A set is strong measure zero if and only for every meagre set M, $X + M \neq \mathbb{R}$.

Topological Erdős universality problem

Observation: A set A is topologically non-universal if and only if there exists a meager set M such that for all $r > 0$,

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(Reason: ($\Longleftarrow$) $\exists$ meager M such that $A \cap (rM + t) \neq \emptyset$ $\forall$r$\neq 0$ and $t \in \mathbb{R}$)

Topological Erdős universality problem

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Theorem (Jung and Lai, 2023 in preparation)

A subset A of $\mathbb{R}$ is topologically universal if and only if A is a set of strong measure zero.

Topological Erdős universality problem

Conjecture (Borel Conjecture)

There is no uncountable strong measure zero sets.

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Theorem

BC *is independent of* **ZFC**.

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Theorem (Jung and L.)

Existence of uncountable topologically universal set is independent of **ZFC***.*

Topological Erdős universality problem

A set A is full-measure non-universal if and only if there exists a meager set M, with $\mathcal{L}(M) = 0$ such that for all $r > 0$,

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Theorem (Erdős-Kunen-Mauldin, 1981)

Let C be a Cantor set on $\mathbb{R}^1$. Then there exists a Cantor set M with $m(M) = 0$ such that $C + M = \mathbb{R}$.

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The End!

Thank you for your time!

A Week in the Life of a Mathematician

(with apologies to Michael Flanders and Donald Swann)

'Twas on a Monday morning I had a bright idea,
I was lying in the bath tub and the strategy seemed clear,
For a problem posed by Erdös back in nineteen forty nine,
On sequences dilated into subsets of the line

'Twas on a Tuesday morning I jotted down my thoughts,
I covered backs of envelopes with surds and aleph noughts.
After several cups of coffee I began to feel inspired,
And a lengthy calculation gave the answer I desired.

'Twas on a Wednesday morning I wrote the details out.
My lemmas and corollaries left little room for doubt.
I filled up many pages just to get the logic right,
And with epsilons and deltas I made it watertight.

'Twas on a Thursday morning I typed the paper up,
With "slash subset" and "slash mapsto" to say nothing of "slash cup".
My LaTeXing was perfect, printed out it looked so good,
Should I send it to the *Annals*? I rather thought I would!

'Twas on a Friday morning I read the paper through,
I checked out every detail as good authors ought to do.
At the bottom of page twenty in an integral I found,
I'd divided through by zero and the proof crashed to the ground.

On Saturday and Sunday I was too depressed to care,
So 'twas on a Monday morning that I had my next idea.

KJF