Birth-death type random walks on hyperbolic graphs

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Dedicated to Professor Ka-Sing Lau

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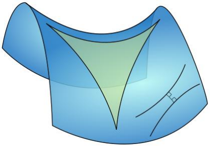
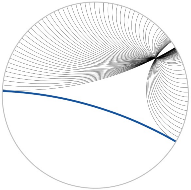
What is a “hyperbolic” space?

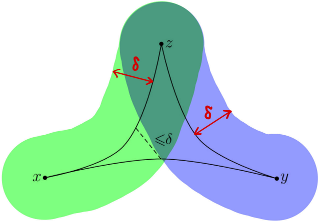


Saddle surface



Poincaré disk

Observed from classic hyperbolic geometry, a geodesic metric space is said to be hyperbolic if *every geodesic triangle is “δ-thin” for some global constant $\delta \geq 0$.*



(Rips’) thin triangle property

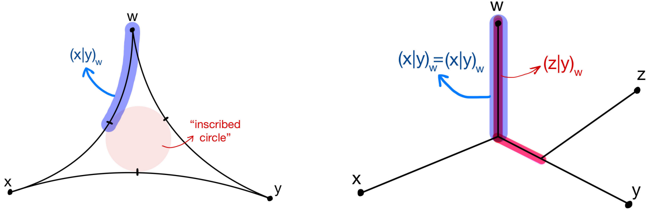
(Gromov-)hyperbolic graph

Let $\mathcal{G} = (X, E)$ be a connected infinite graph with graph distance d.

According to [Gromov, 'Essays in Group Theory', 1987], we say that $\mathcal{G}$ is Gromov-hyperbolic (or simply, hyperbolic) if $\exists\ \delta \geq 0$ such that

$$(x|y)_w \geq \min\{(x|z)_w,\ (z|y)_w\} - \delta, \quad \forall\ w, x, y, z \in X.$$

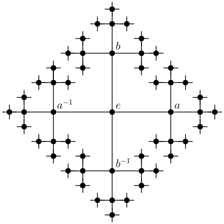
where $(x|y)_w := \frac{d(w,x)+d(w,y)-d(x,y)}{2}$ is the Gromov product.



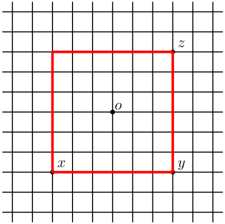
- Gromov-hyperbolicity $\Leftrightarrow$ thin triangle property (with different δ).
- Any tree is hyperbolic (with $\delta = 0$).

Hyperbolic group

In geometric group theory, a finitely generated group Γ is called hyperbolic if its Cayley graph (Γ, E_S) is hyperbolic.



Free groups are hyperbolic



$\mathbb{Z}^n$ is not hyperbolic for $n \geq 2$

Although there is a wealth of remarkable work on hyperbolic groups, in this study our focus is on hyperbolic graphs without group structures (see the *examples* below).

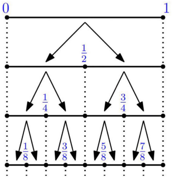
Gromov boundary

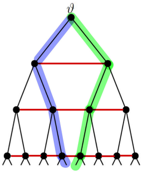
Let $\mathcal{G} = (X, E)$ be a locally finite hyperbolic graph. Fix a root $\vartheta \in X$, and write $(x|y)$ for $(x|y)_\vartheta$. For small $a > 0$, $\exists$ visual metric ϱ_a on X:

$$\varrho_a(x, y) \asymp e^{-a(x|y)}, \quad \forall\, x, y \in X,\, x \neq y.$$

The Gromov boundary $\partial\mathcal{G} := (\varrho_a\text{-completion of } X) \setminus X$.

- $\partial\mathcal{G} =$ all "asymptotic classes" of geodesic rays from ϑ.
- ∂(binary tree) $\approx$ triadic Cantor set.

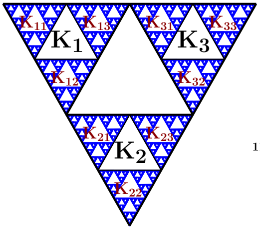


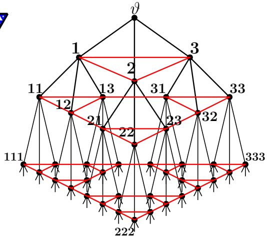


Example: binary partition on $[0, 1]$

On a binary tree, add "horizontal edges" when two subintervals are intersected. This $\mathcal{G}$ is hyperbolic, and $\partial\mathcal{G} \approx [0, 1]$.

Example: augmented trees





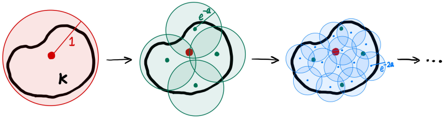
[Kaimanovich, 2003]
The Sierpiński graph
$\mathcal{G} = (\Sigma^*, E_v \cup E_h)$ is
based on the ***symbolic tree*** (Σ^*, E_v) of SG.

It is hyperbolic, and
$\partial\mathcal{G} \approx \mathrm{SG}$.

- [Lau-Wang, 2009]: extend the study of augmented *(symbolic)* tree $\mathcal{G}$ to self-similar sets with open set condition (OSC).
- [Luo-Lau, 2013] etc: bi-Lipschitz classification of self-similar sets via their augmented trees.
- [Lau-Wang, 2017]: a relaxation of E_h; $\mathcal{G}$ has bounded degree $\Leftrightarrow$ OSC.
- [K.-Lau-Wong, 2017]: "natural" (reversible) random walks on $\mathcal{G}$ and induced (quadratic) forms on self-similar sets.

Example: hyperbolic filling [Bourdon-Pajot, 03] etc

Let K be a compact metric space. Fix $a > 0$ and $\gamma > 1$. For each $m \geq 0$, let X_m be a maximal e^{-am}-separated set ("net") in K.



Set $B_x := B(x, \gamma e^{-am})$ for $x \in X_m$. We call $\mathcal{G} = (X, E_v \cup E_h)$ with

- $X = \coprod_{m \geq 0} X_m$,
- $E_v \subset \cup_{m \geq 0}\{(x, y) \in X_m \times X_{m+1} : B_x \cap B_y \neq \emptyset\}$, and
- $E_h = \cup_{m \geq 1}\{(x, y) \in X_m \times X_m : B_x \cap B_y \neq \emptyset\}$

a hyperbolic filling of K. It is hyperbolic, and $\partial \mathcal{G} \approx K$.

Some remarkable applications of hyperbolic fillings include:

- on conformal dimension: [Carrasco Piaggio, 2013] etc.
- extension/trace results of Sobolev spaces on metric spaces: [Bonk-Saksman, 2018], [Björn-Björn-Shanmugalingam, 2022] etc.
- on conformal walk dimension: [Kajino-Murugan, 2023].

In summary, our main concern is the hyperbolic (rooted) graph $\mathcal{G}$ arising from partition/iteration/filling... in a fractal/metric space K. (Every one here has a belief on what is a fractal.) In such $\mathcal{G}$,

- each vertex $\to$ a (pre-)compact subset of K (“position”);
- different levels $\to$ different sizes of subsets of K (“scaling”);
- $\partial\mathcal{G} \approx K$.

Our target is to study random walks on such $\mathcal{G}$. However, as K and its family of subsets are quite flexible and $\mathcal{G}$ is lack of symmetry (compared with hyperbolic groups), it seems hopeless to deal with a broad setting.

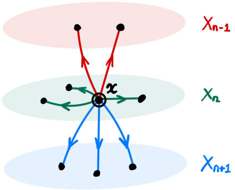
Nevertheless, we are able to investigate a certain class of reversible random walks on general hyperbolic rooted graphs.

Birth-death type random walk

From now on, let $\mathcal{G} = (X, E)$ be a hyperbolic rooted graph. We say that $\{Z_n\}_{n\geq 0}$ is a birth-death type random walk (BDRW) on $\mathcal{G}$ if it satisifies

- reversibility: $\mathbf{m}(x)P(x,y) = \mathbf{c}(x,y) = \mathbf{m}(y)P(y,x)$ ("network"),
- bounded range: $R := \sup\{d(x,y) : P(x,y) > 0\} < +\infty$,
- uniform irreducibility: $\exists\, n_0 > 0$ and $\varepsilon_0 \in (0,1)$ such that $(x,y) \in E \;\Rightarrow\; \exists\, n \leq n_0$ with $P^n(x,y) \geq \varepsilon_0$,

and the "birth-death property":



$\mathbb{P}\big(\big||Z_{n+1}| - |Z_n|\big| \leq 1\big) = 1$ (where $|x| := d(\vartheta, x)$), and $\exists\, \boldsymbol{\lambda} = (\lambda_i)_{i\geq 1}$ such that

$$\frac{\mathbb{P}(|Z_{n+1}| = |x| - 1 \mid Z_n = x)}{\mathbb{P}(|Z_{n+1}| = |x| + 1 \mid Z_n = x)} = \lambda_{|x|}, \;\forall\, x \in X.$$

We also call $\{Z_n\}_{n\geq 0}$ a $\boldsymbol{\lambda}$-BDRW.

Basic facts of BDRWs

Suppose $\{Z_n\}_{n\geq 0}$ is a $\boldsymbol{\lambda}$-BDRW on $\mathcal{G}$. Then

- $\mathcal{G}$ has bounded degree, i.e., $\sup_{x\in X} \deg(x) < +\infty$.
- $0 < C^{-1} \leq \lambda_i \leq C < \infty$, $\forall\, i \geq 1$.
- Define a sequence of stopping times $\{t_j\}_{j=0}^{\infty}$ by letting $t_0 = 0$ and $t_j = \inf\{n \geq t_{j-1} : |Z_n| \neq |Z_{t_{j-1}}|\}$ for $j \geq 1$. Then $\{|Z_{t_j}|\}_{j\geq 0}$ (called the level process of $\{Z_n\}_{n\geq 0}$) is a birth-death chain with transition probabilities $Q(0,1) = 1$, $Q(i, i-1) = \frac{\lambda_i}{\lambda_i - 1}$ and $Q(i, i+1) = \frac{1}{\lambda_i + 1}$.
- $\{Z_n\}_{n\geq 0}$ is transient if and only if its level process is transient (i.e., $\sum_{m=1}^{\infty} \lambda_1\lambda_2\cdots\lambda_m < \infty$).

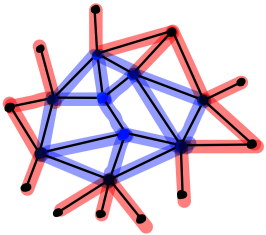
Martin boundary & hitting distribution

Now suppose $\{Z_n\}_{n\geq 0}$ is transient.

- Green function $G(x,y) = \sum_{n=0}^{\infty} P^n(x,y) < \infty$, and
- Martin kernel $K(x,y) = \frac{G(x,y)}{G(\vartheta,y)}, \ x,y \in X$.
- Let $\widehat{X}_M$ be the minimal compactification of X on which every $K(x,\cdot), x \in X$ is continuously extended.
- The Martin boundary $\mathcal{M} = \widehat{X}_M \setminus X$.
- Under the topology of $\widehat{X}_M$, $Z_n \to Z_\infty$ (an $\mathcal{M}$-valued r.v.) a.s.
- Hitting distribution $\nu_x(\cdot) = \mathbb{P}_x(Z_\infty \in \cdot), \ x \in X$.
- Let $\nu := \nu_\vartheta$. It is known that $\nu_x \ll \nu$, and $\frac{d\nu_x}{d\nu} = K(x,\cdot)$.

Strong isoperimetry of BDRW

- spectral radius: $\mathbf{r}(P) = \limsup_{n\to\infty}(P^n(x,y))^{1/n} \in (0,1]$.



For reversible RW, $\mathbf{r}(P) < 1$ if and only if it has strong isoperimetry, i.e.,

$$\sup\left\{\frac{\mathbf{m}(F)}{\mathbf{c}(F, X\setminus F)} : \text{finite } F \subset X\right\} < \infty.$$

($\mathbf{m}(F)$ = "volume" of F, $\mathbf{c}(F, X\setminus F)$ = "surface area" of F.)

Theorem [K., 2023+]. $\boldsymbol{\lambda}$-BDRW $\{Z_n\}_{n\geq 0}$ has strong isoperimetry if and only if its level process has strong isoperimetry, i.e.,

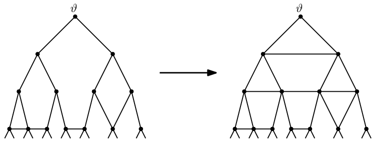
$$\sup_{m\geq 1}\sum_{i=1}^{m} \lambda_i\lambda_{i+1}\cdots\lambda_m < \infty.$$

In this case, by Ancona's Theorem, the Martin boundary $\mathcal{M}$ of $\{Z_n\}_{n\geq 0}$ is homeomorphic to $\partial\mathcal{G}$.

Hitting distribution of BDRW (part I)

Now suppose $\{Z_n\}_{n\geq 0}$ has strong isoperimetry. To calculate the hitting distribution ν on $\mathcal{M}$ ($\approx \partial\mathcal{G}$), we use:

- $\mathbb{P}_\vartheta(Z_{\tau_m} = x) = \frac{\mathbf{c}(x, X_{m+1})}{\mathbf{c}(X_m, X_{m+1})}$ by the reversibility, where $x \in X_m$ and τ_m is the (first) hitting time of X_m.
- P-admissible graph $\mathcal{G}^P = (X, E^P)$ with $E^P = \{(x,y) : P(x,y) > 0\}$ is near-isometric to $\mathcal{G}$ (i.e. $\sup_{x,y} |d^P(x,y) - d(x,y)| < \infty$).
- "Expansion" technique [K.-Lau-Wang, unpublished draft, 2021]:



A hyperbolic rooted graph is near-isometric to its expansive hull ("blow wind from bottom" to let all geodesics convex).

This technique is also used in Naïm kernel estimate.

Hitting distribution of BDRW (part II)

Equip the Gromov boundary $\partial\mathcal{G}$ with a visual metric ϱ_a. For $x \in X$, let

$$\mathcal{J}_\partial(x) := \{\xi \in \partial\mathcal{G} : \exists \text{ } P\text{-admissible ray passing through } x \to \xi\},$$

where a "P-admissible ray" is $\{x_i\}_{i=0}^\infty$ with $x_i \in X_i$ and $P(x_i, x_{i+1}) > 0$.

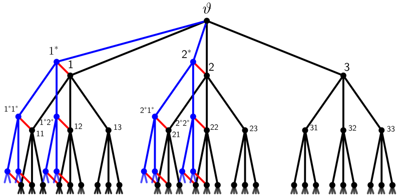
Theorem [K., 2023+]. Suppose $\{Z_n\}_{n\geq 0}$ is a $\boldsymbol{\lambda}$-BDRW on $\mathcal{G}$ with strong isoperimetry (thus $\partial\mathcal{G} \approx \mathcal{M}$). Then the hitting distribution ν satisfies the estimate

$$\nu\big(B_{\varrho_a}(\xi, e^{-a|x|})\big) \asymp \frac{\mathbf{m}(x)}{\mathbf{c}(X_{|x|}, X_{|x|+1})}, \quad \forall\, x \in X,\ \xi \in \mathcal{J}_\partial(x).$$

Consequently, ν is a volume doubling measure on $(\partial\mathcal{G}, \varrho_a)$.

Intermezzo: Existence of BDRWs

Q. Does a BDRW with range $R = 1$ (i.e., of neighbor-type) always exist?



The answer is No!
A counterexample is the graph $\mathcal{G} = T_3 \sqcup T_2^*$ with T_2^* "pairwise embedded" into T_3.

We should allow mass transfer from a vertex to its "close relatives".

It follows from a similar argument as in [Vol'berg-Konyagin, 1987] (to construct volume doubling measures on doubling metric spaces) that

Theorem [K., 2023+]. A hyperbolic graph $\mathcal{G}$ carries a BDRW if and only if $\mathcal{G}$ is roughly starlike (i.e., $\sup_{x \in X} \inf_{\text{ray } \varpi \text{ in } X} \operatorname{dist}(x, \varpi) < \infty$) and has bounded degree.

Naïm's Θ-kernel

- First define the Naïm kernel on $X \times X$ by

$$\Theta(x,y) = \frac{K(x,y)}{G(x,\vartheta)} = \frac{F(x,y)}{F(x,\vartheta)G(\vartheta,\vartheta)F(\vartheta,y)}, \quad x,y \in X.$$

- It can be continuously extended to $X \times \mathcal{M}$ as $K(\cdot,\cdot)$ does.
- The extension of $\Theta(\cdot,\cdot)$ on $(\mathcal{M} \times \mathcal{M}) \setminus \mathrm{diag}$ is more involved:
  - ξ-process $\{Z_n^\xi\}_{n=0}^\infty$: $P_\xi(x,y) = P(x,y)\frac{K(y,\xi)}{K(x,\xi)}$.
  - The corresponding hitting distribution $\nu^\xi = \delta_\xi$.
  - Fix a sequence $\{Y_j\}_{j=0}^\infty$ of finite sets with $Y_j \nearrow X$. Define

$$\Theta(\xi,\eta) = \lim_{j\to\infty} \sum_{z\in X} \ell_j^\xi(z)\Theta(z,\eta), \quad \xi,\eta \in \mathcal{M},\ \xi \neq \eta,$$

    where $\ell_j^\xi(z)$ is the exit probability of Y_j at z for the ξ-process.
  - The above limit exists since the sum is increasing in j.

Naïm kernel estimate of BDRW

Weak exponential condition: $\exists\ C > 0$ and $q \in (0,1)$ such that

$$\lambda_m \lambda_{m+1} \cdots \lambda_{m+\ell-1} \leq C q^{\ell}, \quad \forall\ m, \ell \geq 1.$$

This condition $\Rightarrow$ $\boldsymbol{\lambda}$-BDRW has strong isoperimetry $\Rightarrow$ transience.

Theorem [K., 2023+]. Suppose $\{Z_n\}_{n\geq 0}$ is a $\boldsymbol{\lambda}$-BDRW on $\mathcal{G}$, and $\boldsymbol{\lambda}$ satisfies the weak exponential condition. Then the Naïm's Θ-kernel satisfies the estimate

$$\Theta(\xi,\eta) \asymp \frac{1}{\Lambda_{(\xi|\eta)} V(\xi,\eta)}, \quad \forall\ \xi,\eta \in \partial\mathcal{G}(\approx \mathcal{M}),\ \xi \neq \eta,$$

where $(\xi|\eta) := \inf\{\liminf_{i\to\infty}(x_i|y_i) : x_i \to \xi,\ y_i \to \eta\}$, $\Lambda_0 := 1$, $\Lambda_{(\xi|\eta)} := \lambda_1\lambda_2\cdots\lambda_{\lceil(\xi|\eta)\rceil}$ for $(\xi|\eta) > 0$, $V(\xi,\eta) := \nu\big(B_{\varrho_a}(\xi, \varrho_a(\xi,\eta))\big)$.

Induced (quadratic) form on Martin boundary

- For $u \in L^1(\mathcal{M}, \nu)$, the Poisson integral Hu is given by

$$(Hu)(x) = \mathbb{E}_x u(Z_\infty) = \int_{\mathcal{M}} K(x, \xi) u(\xi) d\nu(\xi), \quad x \in X.$$

- In a network $(X, \mathbf{c})$, a function $f : X \to \mathbb{R}$ has the discrete energy

$$\mathcal{E}_X(f) = \frac{1}{2} \sum_{x,y \in X} \mathbf{c}(x, y) |f(x) - f(y)|^2.$$

- Together with H defines an "induced energy" of $u \in L^2(\mathcal{M}, \nu)$:

$$\mathcal{E}_{\mathcal{M}}(u) := \mathcal{E}_X(Hu) = \frac{1}{2} \sum_{x,y \in X} \mathbf{c}(x, y) |Hu(x) - Hu(y)|^2.$$

- Silverstein's formula [1974] for induced (quadratic) form on $\mathcal{M}$:

$$\mathcal{E}_{\mathcal{M}}(u) = \frac{\mathbf{m}(\vartheta)}{2} \iint_{\mathcal{M} \times \mathcal{M}} |u(\xi) - u(\eta)|^2 \Theta(\xi, \eta) d\nu(\eta) d\nu(\xi).$$

Induced form of BDRW

Whenever $\boldsymbol{\lambda}$ satisfies the weak exponential condition, the induced form of $\boldsymbol{\lambda}$-BDRW on a hyperbolic graph $\mathcal{G}$ satisfies

$$\mathcal{E}_{\partial\mathcal{G}}(u) = \mathcal{E}_X(Hu) \asymp \iint_{\partial\mathcal{G}\times\partial\mathcal{G}} \frac{|u(\xi)-u(\eta)|^2}{\Lambda_{(\xi|\eta)}V(\xi,\eta)}\, d\nu(\eta)d\nu(\xi).$$

In case that $\lambda_i \equiv \lambda \in (0,1)$, we have

$$\Lambda_{(\xi|\eta)} \asymp \varrho_a^{\beta}(\xi,\eta) \quad \text{with } \beta = |\log\lambda|/a.$$

By changing $\boldsymbol{\lambda} = \{\lambda_i\}_{i=1}^{\infty}$ appropriately, we can obtain for some non-decreasing positive function Ψ,

$$\Lambda_{(\xi|\eta)} \asymp \varrho_a^{\beta}(\xi,\eta)\Psi(-\log\varrho_a(\xi,\eta)) \quad \text{with } \beta = |\log\lambda|/a.$$

Revisit: random walks on augmented trees

In [K.-Lau-Wong, 2017], we constructed a BDRW with $\lambda_i \equiv \lambda \in (0,1)$ (called λ-natural random walk there) on the augmented tree $\mathcal{G}$ of a self-similar set K satisfying OSC, and proved that

$$\mathcal{E}_K(u) \asymp \iint_{K\times K} \frac{|u(\xi)-u(\eta)|^2}{|\xi-\eta|^{\alpha+\beta}} d\mathcal{H}^\alpha(\eta) d\mathcal{H}^\alpha(\xi),$$

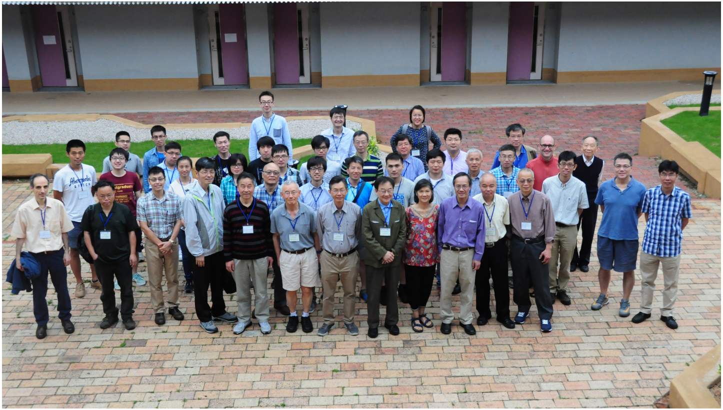
where $\alpha = \dim_H K$ and $\beta = \frac{\log \lambda}{\log r}$.

By changing $\boldsymbol{\lambda} = \{\lambda_i\}_{i=1}^\infty$ appropriately, we can obtain for some non-decreasing positive function Ψ, the $\boldsymbol{\lambda}$-BDRW on $\mathcal{G}$ satisfies

$$\mathcal{E}_K(u) \asymp \iint_{K\times K} \frac{|u(\xi)-u(\eta)|^2}{|\xi-\eta|^{\alpha+\beta}\Psi(-\log|\xi-\eta|)} d\mathcal{H}^\alpha(\eta) d\mathcal{H}^\alpha(\xi).$$

Q. What happens when $\beta = \beta^*$ (= walk dimension of K)?

Thank you very much for listening!



A group photo of Ka-Sing's academic family in 2018