On the fibres of planar self-similar sets with dense rotations

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Known results on the size of fibres of planar self-similar sets with dense rotations

Marstrand's projection and section theorem

$\mathcal{H}^s$: s-dimensional Hausdorff measure.
$\mathcal{L}$: one-dimensional Lebesgue measure.
$\ell_{x,\theta}$: line in the complex plane $\mathbb{C}$ passing through x in direction θ.
Proj_θ: orthogonal projection from $\mathbb{C}$ to $\ell_{0,\theta}$.

Theorem (Marstrand 1954)

Take $1 < s \leq 2$. Let $E \subset \mathbb{C}$ be a s-set, i.e. $0 < \mathcal{H}^s(E) < \infty$. Then

1. $\mathcal{L}(\mathrm{Proj}_\theta(E)) > 0$ *for Lebesgue almost every* $\theta \in [0, 2\pi)$.
2. $\dim_H E \cap \ell_{x,\theta} = s - 1$ *and* $\mathcal{H}^{s-1}(E \cap \ell_{x,\theta}) < \infty$ *for* $\mathcal{H}^s|_E$*-a.e.* $x \in E$ *and Lebesgue almost every* $\theta \in [0, 2\pi)$.

Implication: For Lebesgue almost every $\theta \in [0, 2\pi)$,

$$s - 1 + \dim_H\{y : \dim_H E \cap \mathrm{Proj}_\theta^{-1}(y) \geq s - 1\} = s,$$

that is, E is dimension conserving in direction θ.

Furstenberg's dimension conservation

Let Q be the unit square in $\mathbb{C}$ and $A \subset Q$ be a closed set.

- *mini-set*: a closed set $A' \subset Q$ is a mini-set of A if there exist $\rho > 0$ and $a \in \mathbb{C}$ such that $A' \subset (\rho A + a) \cap Q$.
- *micro set*: a closed set $A'' \subset Q$ is a micro set of A if there exists a sequence A'_n of mini-sets of A such that $A'_n \to A''$ in Hausdorff metric.

Definition

A set A is called *homogenous* if every micro set of A is a mini-set of A.

Theorem (Furstenberg 2008)

If A is homogenous, then A is dimension conserving in **every** *direction.*

Let

$$\mathcal{I} = \{f_k(x) = \rho_k e^{i\lambda_k} x + a_k\}_{k=1}^{n}$$

be an iterated function system (IFS) in $\mathbb{C}$ with contraction $\rho_k \in (0, 1)$, rotation $\lambda_k \in [0, 2\pi)$ and translations $a_k \in \mathbb{C}$.

There exists a unique compact set $A \subset \mathbb{C}$ such that

$$A = \bigcup_{k=1}^{n} f_k(A).$$

We call A the self-similar set of $\mathcal{I}$. We say that A or $\mathcal{I}$ satisfies

- the strong separation condition (SSC) if $f_k(A)$, $k = 1, \cdots, n$ are disjoint;
- the open set condition (OSC) if there exists an open set O such that $O \supset \cup_{k=1}^{n} f_k(O)$ and the sets in the union are disjoint.

Corollary (Furstenberg 2008)

If $\lambda_k = 0$ for $k = 1, \cdots, n$ and $\mathcal{I}$ satisfies SSC, then the self-similar set A of $\mathcal{I}$ is homogenous. Consequently, A is dimension conserving in every direction.

Theorem (Falconer and J. 2014)

If $\lambda_k \in \pi\mathbb{Q}$ for $k = 1, \cdots, n$, then A is dimension conserving in every direction.

- Follow [Feng and Hu 2009] to prove dimension conservation holds for self-similar measures in every direction.

Definition

If there exists $\lambda_k \notin \pi\mathbb{Q}$, then we say that A has dense rotations.

Question

Are planar self-similar sets with dense rotations dimension conserving in every direction?

- Almost true in ‘most’ directions.
- Not true for all self-similar measures.
- True for ‘most’ self-similar measures.
- No counter-examples for self-similar sets yet.

Theorem (Falconer and J. 2015)

Let A be a planar self-similar set with dense rotations. Suppose that ~~$\mathcal{I}$ satisfies OSC and~~ $\dim_H A = s > 1$. Then there exists a set $\mathcal{E} \subset [0, 2\pi)$ with $\dim_H \mathcal{E} = 0$ such that for every $\theta \in [0, 2\pi) \setminus \mathcal{E}$,

$$\mathcal{L}\{y : \dim_H A \cap \mathrm{Proj}_\theta^{-1}(y) \geq s - 1 - \epsilon\} > 0$$

for every $\epsilon > 0$.

- follow the idea of [Shmerkin and Solomyak 2015] to use Erdös-Kahane type arguments to prove absolute continuity of projected random measures in 'most' directions;
- then use percolation method to provide a lower bound of the dimension of fibre sets.

Write $\rho e^{i\lambda} \in \mathbb{D} = \{|z| < 1\}$ and denote by

$$\mathcal{I}_{\rho,\lambda,\bar{a}} = \{f_k(x) = \rho e^{i\lambda}x + a_k\}_{k=1}^n$$

the IFS with contraction ratio ρ, rotation λ and translation $\bar{a} = (a_k)_{k=1}^n \in \mathbb{C}^n$.

Theorem (Rapaport 2017)

There exist $\rho e^{i\lambda} \in \mathbb{D}$ and $\bar{a} \in \mathbb{C}^n$ such that $\mathcal{I}_{\rho,\lambda,\bar{a}}$ satisfies SSC, its associated uniform self-similar measure μ has $\dim_H \mu = s > 1$, and there exists a G_δ dense subset $B \subset [0, 2\pi)$ such that for $\theta \in B$,

1. *$\mathrm{Proj}_\theta^*(\mu)$ is singular (but still has Hausdorff dimension 1);*
2. *the fibre measures $\mu_{y,\theta}$ are discrete (so in particular has Hausdorff dimension 0) for $\mathrm{Proj}_\theta^*(\mu)$-a.e. y.*

Counter-examples for self-similar measures

The point $\rho e^{i\lambda} \in \mathbb{D}$ and translation $\bar{a}$ in the counter-example are picked as follows.

- $(\rho e^{i\lambda})^{-1}$ is a 'complex Pisot number', i.e., an algebraic number such that all its Galois conjugates, apart from $\rho^{-1}e^{i\lambda}$, are in $\mathbb{D}$;
- the minimal polynomial of $(\rho e^{i\lambda})^{-1}$ has constant term 1 or -1;
- $\lambda \notin \pi\mathbb{Q}$ and $\rho \in (\frac{1}{4}, \frac{1}{3})$;
- $n = 4$ and a_k, $k = 1, 2, 3, 4$, are in the forms of $\pm m(\rho e^{i\theta})^l$ for two pairs of $m, l \in \mathbb{N}$ such that $\mathcal{I}_{\rho,\lambda,\bar{a}}$ satisfies SSC.

The idea of proof is to provide constant lower bounds for the modulus of Fourier transforms of projected measures in an open dense set of directions since there are infinitely many $\mathrm{Re}((\rho e^{i\lambda})^{-N})$ that are really close to integers.

True for 'most' self-similar measures

Theorem (Rapaport 2020)

There exists a set $\mathcal{E} \subset \mathbb{D}$ *with* $\dim_H \mathcal{E} = 0$ *such that for* $\rho e^{i\lambda} \in \mathbb{D} \setminus \mathcal{E}$ *with* $\lambda \notin \pi\mathbb{Q}$ *the following holds. If* $\bar{a} \in \mathbb{C}^n$ *is such that* $\mathcal{I}_{\rho,\lambda,\bar{a}}$ *satisfies SSC and its associated self-similar measure* μ *has* $\dim_H \mu = s > 1$*, then for every* $\theta \in [0, 2\pi)$*,*

1. $\mathrm{Proj}_\theta^*(\mu)$ *is absolutely continuous;*
2. *the fibre measures* $\mu_{y,\theta}$ *are exact-dimensional with dimension* $s - 1$ *for* $\mathrm{Proj}_\theta^*(\mu)$*-a.e.* y*.*

- Use [Shmerkin and Solomyak 2015] and [Shmerkin 2019] to show the Radon-Nikodym derivatives $\mathrm{d}\mathrm{Proj}_\theta^*(\mu)/\mathrm{d}\mathcal{L}$ is uniformly bounded in $L^q(\mathcal{L})$ for some $q > 1$ for all θ, then use Borel-Cantelli to estimate local dimension of fibre measures.

None of the results so far can say anything about the size of fibres for a given self-similar set with dense rotations in a given direction. The only precise result that I know is the following one using percolation method, but only for box-counting dimension rather than Hausdorff dimension.

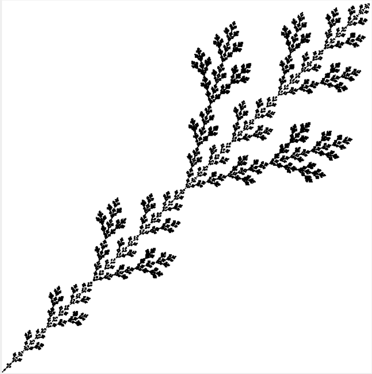
Theorem (Falconer and J. 2015)

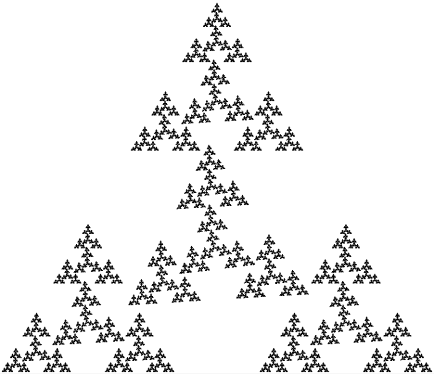
Let A be a planar self-similar set with dense rotations. Suppose that $\mathcal{I}$ satisfies OSC, $\dim_H A = s > 1$ and $\mathrm{Proj}_\theta(A)$ is an **interval** *for all $\theta \in [0, 2\pi)$. Then for every $\theta \in [0, 2\pi)$,*

$$\dim_H\{y : \underline{\dim}_B A \cap \mathrm{Proj}_\theta^{-1}(y) > s - 1 - \epsilon\} = 1$$

for every $\epsilon > 0$.

A connected and a totally disconnected planar self-similar set with dense rotations whose projections are intervals





Some new results using percolation method

Mandelbrot percolation: [Mandelbrot 1972], [Kahane and Peyrière 1976], [Biggins 1977], [Durrett and Liggett 1983], [Kahane 1987], [Chayes, Chayes and Durrett 1988], [Fan 1990], [Lyons 1992], [Waymire and Williams 1995], [Barral 1999] etc.

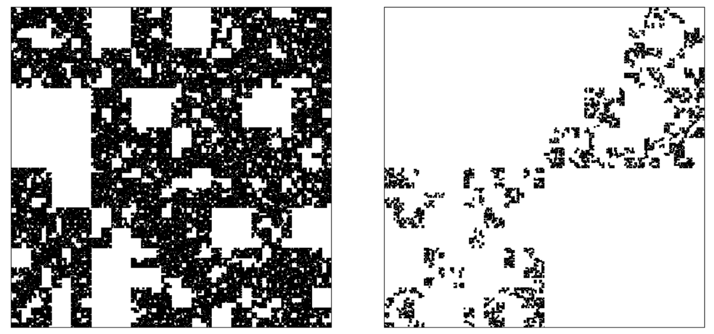


Figure: Picture by [Fraser and Yu 2018]

From now on we always assume A is the self-similar set of an IFS $\mathcal{I}_{\rho,\lambda,\bar{a}}$ for $\rho \in (0,1)$, $\lambda \notin \pi\mathbb{Q}$, $\bar{a} \in \mathbb{C}^n$ and $\mathcal{I}_{\rho,\lambda,\bar{a}}$ satisfies SSC.

For $\sigma \in \{1, \cdots, n\}^l$ write $f_\sigma = f_{\sigma_1} \circ \cdots \circ f_{\sigma_l}$ and $A_\sigma = f_\sigma(A)$.
Fix $\alpha > 0$. To define a ρ^α-Mandelbrot percolation on A:

- $A_\emptyset = A$ survives;
- if A_σ survives, write $A_\sigma = \cup_{k=1} f_\sigma \circ f_k(A) = \cup_{k=1}^{n} A_{\sigma \cdot k}$ and, independently, each $A_{\sigma \cdot k}$ survives with probability ρ^α and killed with probability $1 - \rho^\alpha$;
- define $A_\omega^\alpha = \bigcap_{l \geq 1} \bigcup_{|\sigma| = l,\, A_\sigma \text{ survives}} A_\sigma$.

The Mandelbrot percolation A_ω^α are the points in A that survive indefinitely through this survive/killed process.

There is a natural measure μ_ω^α carried by A_ω^α called Mandelbrot cascade measure. It is defined as the almost sure weak limit of

$$\mu_{\omega,n}^\alpha(\mathrm{d}x) = \rho^{-\alpha n}\mathbf{1}_{\{A_{x,n} \text{ survives}\}}\,\mu(\mathrm{d}x),\ x \in A,$$

where for $n \geq 1$, $A_{x,n}$ denotes the piece of A in generation n that contains x, and μ is the uniform self-similar measure of A.

By [Kahane and Peyrière 1976], if $\alpha < \dim_H A$ then with positive probability,

$$\dim_H A_\omega^\alpha = \dim_H \mu_\omega^\alpha = \dim_H A - \alpha.$$

Radial projection and main result

For $x \in \mathbb{C}$ let

$$\pi_x(z) := \frac{z - x}{|z - x|}, \; z \neq x$$

denote the radial projection from $\mathbb{C}$ towards x.

Theorem (J. 2023)

Let $\dim_H A - \alpha > 1$. If, with positive probability for μ_ω^α-positively many $x \in A_\omega^\alpha$, the radial projection

$$\pi_x(A_\omega^\alpha \setminus A_{x,1}) \text{ contains an interval},$$

then for every $\theta \in [0, 2\pi)$,

$$\dim_H\{y : \dim_H A \cap \mathrm{Proj}_\theta^{-1}(y) \geq \alpha\} = 1.$$

Theorem (Orponen 2019)

If $\mu \in \mathcal{M}(\mathbb{C})$ and $I_s(\mu) = \iint \frac{\mathrm{d}\mu(x)\mathrm{d}\mu(y)}{|x-y|^s} < \infty$ for $s > 1$, then

$$\dim_H\{x \in \mathbb{C} : \pi_x^*(\mu) \text{ is singular }\} \leq 2 - s.$$

Since $\dim_H \mu_\omega^\alpha = \dim_H A - \alpha > 1$, one has

$$2 - (\dim_H A - \alpha) < 1 < \dim_H \mu_\omega^\alpha.$$

Thus Orpopen's theorem implies that for μ_ω^α-almost every $x \in A_\omega^\alpha$,

$$\mathcal{L}(\pi_x(A_\omega^\alpha \setminus A_{x,1})) > 0.$$

Hence our assumption is stipulated from "with positive length" to "exist interior points".

Idea of proofs

The α-capacity of a set E is defined as
$\mathrm{Cap}_\alpha(E) := \sup\{I_\alpha(\mu)^{-1} : \mu \text{ is a Borel prob. measure carried by } E\}$.

Theorem (Lyons 1992)

There exists constants $0 < c_1 < c_2 < \infty$ such that for any analytic set $E \subset A$, $c_1 \mathrm{Cap}_\alpha(E) \le \mathbb{P}(E \cap A_\omega^\alpha \neq \emptyset) \le c_2 \mathrm{Cap}_\alpha(E)$.

Similar results were obtained previously by [Kahane 1987] and [Fan 1990].

Corollary (J. 2023)

Given $\theta \in [0, 2\pi)$, $\dim_H\{y : \dim_H A \cap \mathrm{Proj}_\theta^{-1}(y) \ge \alpha\}$ is no less than

$$\operatorname{ess\,sup}_\omega \mathrm{Proj}_\theta\{x \in A : \ell_{x,\theta} \cap A_\omega^\alpha \text{ contains at least two points}\}.$$

By recurrence arguments one can show that, if with positive probability for μ_ω^α-positively many x, $\pi_x(A_\omega^\alpha \setminus A_{x,1})$ contains an interval, then within **finite** steps $m_{x,\omega}$,

$$\pi_x(A_\omega^\alpha \setminus A_{x,m_{x,\omega}}) \text{ is the whole circle.}$$

This implies that with positive probability for μ_ω^α-positively many $x \in A_\omega^\alpha$, there is at least one more point in $\ell_{x,\theta} \cap A_\omega^\alpha$ in every direction θ. Thus

$$\dim_H\{y : \dim_H A \cap \mathrm{Proj}_\theta^{-1}(y) \geq \alpha\} \geq \operatorname{ess\,sup}_\omega \dim_H \mathrm{Proj}_\theta(\mu_\omega^\alpha)$$

and $\operatorname{ess\,sup}_\omega \dim_H \mathrm{Proj}_\theta(\mu_\omega^\alpha) = \min\{1, \dim_H A - \alpha\} = 1$ for every $\theta \in [0, 2\pi)$ by [Falconer and J. 2014] (following [Hochman and Shmerkin 2012]).

A precise result for Hausdorff dimension

Corollary (J. 2023)

Let A be a planar self-similar set as in [Falconer and J. 2015], that is $\mathcal{I}$ satisfies OSC, $\dim_H A > 1$ and $\mathrm{Proj}_\theta(A)$ is an interval for all $\theta \in [0, 2\pi)$. For every $\theta \in [0, 2\pi)$,

$$\dim_H\{y : \dim_H A \cap \mathrm{Proj}_\theta^{-1}(y) > 0\} = 1.$$

The lower bound 0 can be improved a bit, but I doubt that the bound could be optimal from this line of arguments.

Question

If $\mathrm{Proj}_\theta(A)$ is an interval for all $\theta \in [0, 2\pi)$, what is the critical probability ρ^{α_c} such that for $\alpha < \alpha_c$, with positive probability, $\mathrm{Proj}_\theta(A_\omega^\alpha)$ is an interval for all $\theta \in [0, 2\pi)$?

About the assumption

We conjecture that the assumption in the theorem that “with positive probability for μ_ω^α-positively many $x \in A_\omega^\alpha$, $\pi_x(A_\omega^\alpha \setminus A_{x,1})$ contains an interval" is true as long as $\dim_H A - \alpha > 1$, which would solve the dimension conservation conjecture. But this seems very hard to prove.

To support this conjecture, we recall the result of [Rams and Simon 2015]:

Theorem (Rams and Simon 2015)

For Mandelbrot percolation Q_ω^α on square Q (as the self-similar set of a square lattice IFS), if $\dim_H Q_\omega^\alpha = 2 - \alpha > 1$, then almost surely conditioned on $Q_\omega^\alpha \neq \emptyset$, the radial projection $\pi_x(Q_\omega^\alpha)$ contains an interval for every $x \in \mathbb{C}$.

Question

Let A be a planar self-similar sets with dense rotations and $\dim_H A > 1$. Is there a point $x \in \mathbb{C}$ such that $\pi_x(A)$ contains an interval? If so, what is the size of such points?

Recall by [Orponen 2019] we know the set of points $x \in \mathbb{C}$ such that $\mathcal{L}(\pi_x(E)) = 0$ has Hausdorff dimension less or equal to $2 - \dim_H E$ for any Borel set E with $\dim_H E > 1$. What more can be said when E is a self-similar set?

A question asked by De-Jun Feng

Let A be a planar self-similar set with dense rotations and $\dim_H A > 1$. For every direction θ, is the set of y such that $A \cap \mathrm{Proj}_\theta^{-1}(y)$ contains at least **two** points has Hausdorff dimension 1?

Related to intersection of 'Cantor sets'

For the self-similar set A of $\mathcal{I}_{\rho,\lambda,\bar{a}}$ one can write $A = A' + \rho e^{i\lambda} A'$, where A' is the self-similar set of $\mathcal{I}_{\rho^2,2\lambda,\bar{a}}$. Then for $x \in \mathbb{R}$, $\theta \in \pi_x(A)$ if and only if

$$(x - \mathrm{Proj}_\theta(A')) \cap \rho \mathrm{Proj}_{\theta+\lambda}(A') \neq \emptyset.$$

Palis conjecture: For two Cantor sets C_1 and C_2, if $\mathcal{L}(C_1 + C_2) > 0$ then $C_1 + C_2$ contains an interval.

- [Moreira and Yoccoz 2001] stable intersection of regular Cantor sets;
- [Takahashi 2019] sum of 'perturbed' self-similar sets;
- [Dekking, Simon, Szekely, Szekeres 2023] interior points of sum of random (in translation) Cantor sets;

Follow [Moreira and Yoccoz 2001], [Takahashi 2019] and use [Orponen 2019] we can show:

Theorem (J. 2023)

Let A be the self-similar set of an IFS $\mathcal{I}_{\rho,\lambda,\bar{a}}$ for $\rho \in (0,1)$, $\lambda \notin \pi\mathbb{Q}$, $\bar{a} \in \mathbb{C}^n$, $\dim_H A = s > 1$ and $\mathcal{I}_{\rho,\lambda,\bar{a}}$ satisfies SSC.
For every $N \geq 1$ one can find an integer $m \geq N$ such that for the m-th iteration IFS $\mathcal{I}_{\rho^m,m\lambda,\bar{a}^{(m)}}$ of $\mathcal{I}_{\rho,\lambda,\bar{a}}$, there exists a translation $\bar{b} \in B(\bar{a}^{(m)}, \rho^m)$ in $\mathbb{C}^{mn}$ such that the IFS $\mathcal{I}_{\rho^m,m\lambda,\bar{b}}$ satisfies SSC and the self-similar set $A_{\bar{b}}$ of $\mathcal{I}_{\rho^m,m\lambda,\bar{b}}$ satisfies

$$\pi_x(A_{\bar{b}} \setminus A_{\bar{b},x,1}) \text{ contains an interval}$$

for $\mu_{\bar{b}}$-positively many $x \in A_{\bar{b}}$, where $\mu_{\bar{b}}$ is the uniform self-similar measure of $\mathcal{I}_{\rho^m,m\lambda,\bar{b}}$.

A partial answer to Feng's question

Corollary (J. 2023)

Let A be the self-similar set of an IFS $\mathcal{I}_{\rho,\lambda,\bar{a}}$ for $\rho \in (0,1)$, $\lambda \notin \pi\mathbb{Q}$, $\bar{a} \in \mathbb{C}^n$, $\dim_H A = s > 1$ and $\mathcal{I}_{\rho,\lambda,\bar{a}}$ satisfies SSC. Then one can find another self-similar set $\tilde{A}$ satisfying SSC and arbitrarily close (in the sense of small perturbation in translation) to A such that

$$\dim_H\{y : \tilde{A} \cap \mathrm{Proj}_\theta^{-1}(y) \text{ contains at least two points}\} = 1$$

for every $\theta \in [0, 2\pi)$.

Thanks for your attention!