Some examples of random covering sets

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Joint work with M. Järvenpää, M. Myllyoja and Ö. Stenflo

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- Dimension of $\mathbf{E}(E_n)$?

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- Law of large numbers, Bernoulli 1713: $E_n = A$ for all $n \in \mathbb{N}$ (Cardano 16th century).

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- Upper bounds depending on fine and coarse multifractal spectrum

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- The proof of Ekström and Persson can be adapted to general (r_n) if $\frac{1}{\alpha}$ is replaced by $s_1(r_n)$.

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Our results

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where

$$\overline{\delta} := \mathop{\mu\text{-ess sup}}_{x, \underline{\dim}_{\mathrm{loc}}\,\mu(x) > s_2(r_n)} \frac{\underline{\dim}_{\mathrm{loc}}\,\mu(x)}{\overline{\dim}_{\mathrm{loc}}\,\mu(x)}.$$

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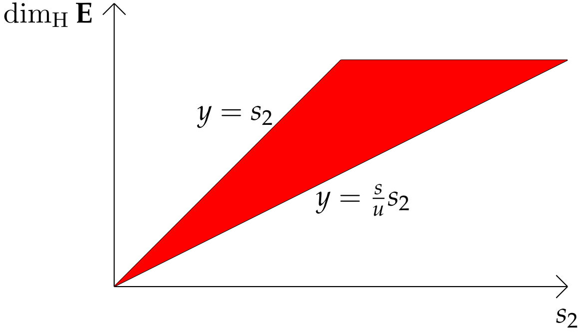
- Given $0 \leq s \leq u \leq d$, there exists μ such that $\underline{\dim}_{\mathrm{loc}}\,\mu(x) = s$ and $\overline{\dim}_{\mathrm{loc}}\,\mu(x) = u$ for all x and for all points $p = (p_1, p_2)$ in the red area there exists (r_n) such that $s_2(r_n) = p_1$ and $\dim_{\mathrm{H}} \mathbf{E}(E_n) = p_2$.



Thank you!