Density of Minimal Points and Bergelson-Hindman question

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Outline

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Background: Recurrence

Theorem (Poincaré Recurrence Theorem)

Let $(X, \mathscr{X}, \mu, T)$ be a m.p.s. Let $A \in \mathscr{X}$ with $\mu(A) > 0$. Then there exists $n \in \mathbb{N}$ such that

$$\mu(A \cap T^{-n}A) > 0.$$

Theorem (Birkhoff Recurrence Theorem)

Let (X, T) be a t.d.s. Then there exists $x \in X$ and a sequence $n_i \nearrow +\infty$ such that

$$T^{n_i}x \to x, \ i \to \infty.$$

Multiple recurrence: $T_1, T_2, \ldots, T_d : X \to X, x \in X, n_i \nearrow +\infty$

$$T_1^{n_i}x \to x, \quad T_2^{n_i}x \to x, \quad \ldots, \quad T_d^{n_i}x \to x, \quad i \to \infty.$$

That is

$$(T_1 \times T_2 \times \ldots \times T_d)^{n_i}(x, x, \ldots, x) \to (x, x, \ldots, x).$$

Minimal points and syndetic sets

Definition

A subset S of $\mathbb{Z}$ is **syndetic** if it has bounded gaps, i.e. there is some $N \in \mathbb{N}$ such that $\{i, i+1, \cdots, i+N\} \cap S \neq \emptyset$ for all $i \in \mathbb{Z}$.

Theorem (Gottschalk-Hedlund, 1955)

Let (X,T) be a t.d.s. Then $x \in X$ is a minimal point if and only if for any neighborhood U of x,

$$N_T(x,U) = \{n \in \mathbb{Z} : T^n x \in U\}$$

is syndetic.

Piecewise syndetic sets: M-system

Definition

A set $S \subset \mathbb{Z}$ is called **thick** if it contains arbitrarily long intervals or, equivalently, if it has nonempty intersection with every syndetic set.

A subset B of $\mathbb{Z}$ is called **piecewise syndetic** if it is the intersection of a syndetic set and a thick set. That is, there is a syndetic subset S such that $\forall L \in \mathbb{N}$, $\exists a_L$ s.t. $a_L + (S \cap [1, L]) \subset B$.

Theorem

Let (X, T) be a t.d.s. $x \in X$. The minimal points are dense in $\overline{\mathrm{Orb}(T, x)}$ if and only if for any neighborhood U

$$N(x, U) = \{n \in \mathbb{Z} : T^n x \in U\}$$

is piecewise syndetic.

In this case, $\overline{\mathrm{Orb}(T, x)}$ is called M-system.

van der Waerden Theorem: Multiple Recurrence

Theorem (Furstenberg-Weiss, *J. Anal. Math.*, 1978)

Let (X,T) be a t.d.s. and $d \in \mathbb{N}$. Then there exists some $x \in X$ and a sequence $n_i \nearrow +\infty$ such that

$$(T \times T^2 \times \ldots \times T^d)^{n_i}(x,x\ldots,x) \to (x,x,\ldots,x), \quad i \to \infty.$$

⋇ If (X,T) is minimal, then the set of multiple recurrent points is residual.

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Theorem (van der Waerden, 1927)

For any $r \in \mathbb{N}$ and any partition $\mathbb{Z} = \bigcup_{i=1}^r C_i$, one of C_i contains arbitrarily long arithmetic progressions.

• van der Waerden: Any piecewise syndetic set contains arbitrarily long arithmetic progressions.

Polynomial van der Waerden theorem

Theorem (Bergelson-Leibman, *JAMS*, 1996)

Let (X,T) be a t.d.s. and $d \in \mathbb{N}$. Suppose $p_1, p_2, \ldots, p_d$ are integral polynomials with $p_j(0) = 0$, $1 \le j \le d$. Then there exists some $x \in X$ and a sequence $n_i \nearrow +\infty$ such that

$$(T^{p_1(n_i)} \times T^{p_2(n_i)} \times \ldots \times T^{p_d(n_i)})(x, x \ldots, x) \to (x, x, \ldots, x), \quad i \to \infty.$$

Theorem (Bergelson-Leibman, *JAMS*, 1996)

Let $r, d \in \mathbb{N}$ and suppose $p_1, p_2, \ldots, p_d$ are integral polynomials with $p_j(0) = 0$, $1 \le j \le d$. For any $r \in \mathbb{N}$ and any partition $\mathbb{Z} = \bigcup_{i=1}^r C_i$, one of C_i contains the form

$$\{m + p_1(n), m + p_2(n), \ldots, m + p_d(n)\},$$

where $m, n \in \mathbb{Z}, n \neq 0$.

Outline

Bergelson-Hindman Question

Question (Bergelson-Hindman, 2001)

Let $B \subseteq \mathbb{Z}$ and $k \in \mathbb{N}$, and let $p_1, p_2, \ldots, p_k$ be integral polynomials with $p_i(0) = 0, 1 \leq i \leq k$. If B is piecewise syndetic in $\mathbb{Z}$, then

$$\{(m,n) \in \mathbb{Z}^2 : m + p_1(n), m + p_2(n), \ldots, m + p_k(n) \in B\}$$

is piecewise syndetic in $\mathbb{Z}^2$.

Theorem (Furstenberg-Glasner, 1998)

Let $B \subseteq \mathbb{Z}$ and $k \in \mathbb{N}$. If B is piecewise syndetic in $\mathbb{Z}$, then

$$\{(m,n) \in \mathbb{Z}^2 : m, m+n, \ldots, m+(k-1)n \in B\}$$

is piecewise syndetic in $\mathbb{Z}^2$.

Piecewise syndetic sets

Definition

A subset S of a countable abelian group $(G,+)$ is **piecewise syndetic** if it is the intersection of a syndetic set and a thick set. That is, there exists a finite subset F of G such that

$$\bigcup_{i\in F}(S-i)$$

is *thick* in G, i.e. it contains a shifted copy of any finite subset of G.

A question by Furstenberg

Furstenberg, *Bulletin of AMS (1981)*

Let (X,T) be a t.d.s. Does there always exist a point x such that

$$(x,x,\ldots,x) \in \Delta_d(X)$$

is a **minimal** point for $T \times T^2 \times \ldots \times T^d$?

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Theorem (Furstenberg, 1978)

Let (X,T) be a distal t.d.s. Then for each point $x \in X$, $(x,x,\ldots,x) \in X^d$ is a minimal point for $T \times T^2 \times \ldots \times T^d$.

A question by Furstenberg

Theorem (H.-Shao-Ye, ISR, 2022)

There is a minimal t.d.s. (X,T) *such that for all* $p \neq q \in \mathbb{N}$ *with* $(p,q)=1$*, and for any* $x \in X$*,* (x,x) *is a transitive but not minimal point of* $T^p \times T^q$*. In particular, this gives a* **negative** *answer to Furstenberg's question!*

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Let $G = SL(2,\mathbb{R})$ and Γ_0 be a lattice of G. For $t,s \in \mathbb{R}$ let

$$h_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad g_s = \begin{pmatrix} e^{-s} & 0 \\ 0 & e^s \end{pmatrix}.$$

Then $(X, \mathscr{B}(X), \mu_0, \{h_t\}_{t\in\mathbb{R}})$ is called the *horocycle flow* on $X = G/\Gamma_0$ and $(X, \mathscr{B}(X), \mu_0, \{g_s\}_{s\in\mathbb{R}})$ is called the *geodesic flow*. If we denote $h = h_1$, then it is well known that $(X, \mathscr{B}(X), \mu_0, h)$ is a strictly ergodic system.

$$g_s h_t g_s^{-1} = h_{e^{-2s}t}, \ \forall t,s \in \mathbb{R}.$$

A question by Furstenberg

Question

Let (X,T) be a minimal t.d.s. and $d \in \mathbb{N}$. Is there a point $x \in X$ such that $(x,x,\dots,x)$ is $T \times T^2 \times \dots \times T^d$-piecewise syndetic recurrent? That is, for any neighborhood U of x,

$$N_{\tau_d}(x^{(d)}, U^d) = \{n \in \mathbb{Z}_+ : (T^n x, T^{2n} x, \dots, T^{dn} x) \in U \times \dots \times U\}$$

is piecewise syndetic.

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Question

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is piecewise syndetic.

Theorem (H.-Shao-Ye, ISR, 2022)

Let (X,T) be a minimal t.d.s. and $d \in \mathbb{N}$. Then for any nonempty open set U of X, there is some $x \in U$ such that

$$N_{\tau_d}(x^{(d)},U^d) = \{n \in \mathbb{Z}_+ : (T^n x, T^{2n} x, \ldots, T^{dn} x) \in U \times \ldots \times U\}$$

is piecewise syndetic.

Question

Problem

Let (X,T) be a t.d.s. and $d \in \mathbb{N}$. Let p_i be integral polynomials with $p_i(0) = 0$, $i = 1, 2, \ldots, d$. Is there a point $x \in X$ such that for any neighborhood U of x, is the set

$$N_{\{p_1,\ldots,p_d\}}(x,U) = \{n \in \mathbb{Z} : (T^{p_1(n)}x, \ldots, T^{p_d(n)}x) \in U \times \ldots \times U\}$$

piecewise syndetic?

Polynomial case

Theorem (H.-Shao-Ye, 2022)

Let (X,T) *be a minimal t.d.s. and* $d \in \mathbb{N}$*. Let* p_i *be integral polynomials with* $p_i(0)=0$*,* $i=1,2,\ldots,d$*. If one of the following conditions is satisfied,*

- $\deg(p_i) \geq 2, 1 \leq i \leq d$*;*
- (X,T) *is weakly mixing;*
- (X,T) *is distal,*

then there is a dense G_δ *set* X_0 *such that for each* $x \in X_0$ *and each neighbourhood* U *of* x

$$N_{\{p_1,\ldots,p_k\}}(x,U) = \{n \in \mathbb{Z} : (T^{p_1(n)}x, \ldots, T^{p_d(n)}x) \in U \times \ldots \times U\}$$

is piecewise syndetic.

Polynomial case

Theorem (H.-Shao-Ye, 2022)

Let (X,T) *be a minimal t.d.s. and* $d \in \mathbb{N}$*. Let* $p_1, \ldots, p_d$ *be integral polynomials such that* $p_i(0) = 0$*,* $1 \le i \le d$*. Then for each open subset* U *there is* $x \in U$ *such that*

$$N_{\{p_1,\ldots,p_d\}}(x,U) = \{n \in \mathbb{Z} : (T^{p_1(n)}x, \ldots, T^{p_d(n)}x) \in U \times \ldots \times U\}$$

is piecewise syndetic.

A combinatorial consequence

Theorem (H.-Shao-Ye, 2022)

Let $d \in \mathbb{N}$ and p_i be an integral polynomial with $p_i(0) = 0$, $1 \leq i \leq d$. If S is piecewise syndetic in $\mathbb{Z}$, then there is is piecewise syndetic subset $A \subseteq \mathbb{Z}$ such that for any $N \in \mathbb{N}$, there is some $a_N \in \mathbb{Z}$ with

$$A \cap [-N, N] \subseteq \{n \in \mathbb{Z} : a_N + p_1(n), a_N + p_2(n), \ldots, a_N + p_d(n) \in S\}.$$

Corollary (Bergelson-Leibman, 1996)

Let $p_1, p_2, \ldots, p_d$ be integral polynomials with $p_i(0) = 0$, $1 \leq i \leq d$. If S is piecewise syndetic in $\mathbb{Z}$, then

$$\{n \in \mathbb{Z} : \exists\ a \in \mathbb{Z} \text{ s.t. } a + p_1(n), a + p_2(n), \ldots, a + p_d(n) \in S\}$$

is piecewise syndetic in $\mathbb{Z}$.

Our results: Polynomial version of FG-Theorem

Theorem (H.-Shao-Ye, 2022)

Let $B \subseteq \mathbb{Z}$ *and* $k \in \mathbb{N}$, *and let* $p_1, p_2, \ldots, p_k$ *be integral polynomials with* $p_i(0) = 0, 1 \le i \le k$. *If* B *is piecewise syndetic in* $\mathbb{Z}$, *then*

$$\{(m,n) \in \mathbb{Z}^2 : m + p_1(n), m + p_2(n), \ldots, m + p_k(n) \in B\}$$

is piecewise syndetic in $\mathbb{Z}^2$.

Theorem (H.-Shao-Ye, 2022)

Let (X,T) *be a minimal t.d.s. and* $k \in \mathbb{N}$. *Let* p_i *be an integral polynomial with* $p_i(0) = 0$, $1 \le i \le k$. *Then for each* $x \in X$ *and each neighbourhood* U *of* x

$$N^{\mathbb{Z}^2}_{\{p_1,\ldots,p_k\}}(x,U) = \{(m,n) \in \mathbb{Z}^2 : T^{m+p_1(n)}x \in U, \ldots, T^{m+p_k(n)}x \in U\}$$

is piecewise syndetic in $\mathbb{Z}^2$.

Outline

Topological dynamical systems induced by polynomials

- Let (X,T) be a t.d.s., $\mathscr{A}=\{n^2\}$ and $x\in X$. Define
  - $\omega_x^{\mathscr{A}}=(T^{n^2}x)_{n\in\mathbb{Z}}=(\ldots,T^{(-1)^2}x,\underset{\bullet}{x},T^{1^2}x,T^{2^2}x,\ldots)\in X^{\mathbb{Z}}$

- $W_x^{\mathscr{A}}=\overline{\mathscr{O}}((T^{n^2}x)_{n\in\mathbb{Z}},\sigma)=\overline{\{(\ldots,T^{(n-1)^2}x,\underset{\bullet}{T}^{n^2}x,T^{(n+1)^2}x,\ldots):n\in\mathbb{Z}\}}\subset X^{\mathbb{Z}}$,

- $$N_\infty(X,\mathscr{A})=\overline{\mathscr{O}}((T^{n^2}x)_{n\in\mathbb{Z}},\langle T^\infty,\sigma\rangle)=\overline{\{(\ldots,T^{m+(n-1)^2}x,\underset{\bullet}{T}^{m+n^2}x,T^{m+(n+1)^2}x,\ldots):n,m\in\mathbb{Z}\}}\subset X^{\mathbb{Z}},$$

where σ is the shift, $T^\infty=\cdots\times T\times T\times\cdots$, and $\langle T^\infty,\sigma\rangle$ is the group generated by T^∞ and σ.

General case

- Let $d \in \mathbb{N}$ and $\mathscr{A} = \{p_1, p_2, \cdots, p_d\}$ be polynomials with $p_i(0) = 0$, $1 \le i \le d$. The point of $(X^d)^{\mathbb{Z}}$ is denoted by

$$\mathbf{x} = (\mathbf{x}_n)_{n\in\mathbb{Z}} = \Big((x_n^{(1)}, x_n^{(2)}, \cdots, x_n^{(d)})\Big)_{n\in\mathbb{Z}} \in (X^d)^{\mathbb{Z}}.$$

- Let $\sigma : (X^d)^{\mathbb{Z}} \to (X^d)^{\mathbb{Z}}$ be the shift map, i.e., for all $(\mathbf{x}_n)_{n\in\mathbb{Z}} \in (X^d)^{\mathbb{Z}}$

$$(\sigma\mathbf{x})_n = \mathbf{x}_{n+1}, \ \forall n \in \mathbb{Z}.$$

- For each $x \in X$, put

$$\omega_x^{\mathscr{A}} \triangleq \big((T^{p_1(n)}x, T^{p_2(n)}x, \ldots, T^{p_d(n)}x)\big)_{n\in\mathbb{Z}} \in (X^d)^{\mathbb{Z}},$$

$$N_\infty(X, \mathscr{A}) = \overline{\bigcup\{\mathscr{O}(\omega_x^{\mathscr{A}}, \sigma) : x \in X\}} \subseteq (X^d)^{\mathbb{Z}}.$$

- Note that $(N_\infty(X, \mathscr{A}), \langle T^\infty, \sigma\rangle)$ is a $\mathbb{Z}^2$-t.d.s.

Condition (♠)

Definition

Let $\mathscr{A} = \{p_1, p_2, \ldots, p_d\}$ be a family of integral polynomials. We say $\mathscr{A}$ satisfies *condition* (♠) if $p_1(0) = \ldots = p_d(0) = 0$ and

1. $p_1(n) = a_1 n, p_2(n) = a_2 n, \ldots, p_s(n) = a_s n$, where $s \geq 0$, and $a_1, a_2, \ldots, a_s$ are distinct non-zero integers;
2. $\deg p_j \geq 2, s+1 \leq j \leq d$;
3. any two of $p_{s+1}, \ldots, p_d$ will not appear in the same sequence $\Big(q(n), q(n+1) - q(1), q(n+2) - q(2), \ldots,\Big)$ for some integral polynomial q with $q(0) = 0$.

An equivalent statement: Density of Minimal points

Theorem

Let (X,T) be a minimal t.d.s. Then the following statements are equivalent:

1. *For any family $\mathscr{A}=\{p_1,p_2,\cdots,p_d\}$ of integral polynomials satisfying $(\spadesuit)$, $(N_\infty(X,\mathscr{A}),\langle T^\infty,\sigma\rangle)$ is an M-system.*
2. *For any integral polynomials $p_1,\ldots,p_d$ with $p_i(0)=0$, $1\le i\le d$, we have that for each $x\in X$ and any neighbourhood U of x*

$$N_{\mathscr{A}}^{\mathbb{Z}^2}(x,U)=\{(m,n)\in\mathbb{Z}^2 : T^{m+p_1(n)}x\in U,\ldots,T^{m+p_d(n)}x\in U\}$$

   is piecewise syndetic in $\mathbb{Z}^2$.

Saturation theorem for polynomials

Theorem (H.-Shao-Ye, 2022 (Glasner-H.-Shao-Weiss-Ye, JEMS))

Let (X,T) *be a minimal t.d.s., and* $\pi : X \to X_\infty$ *be the factor map from* X *to its maximal* ∞*-step pro-nilfactor* X_∞. *Then we have*

$$\begin{array}{ccc} X & \xleftarrow{\;\varsigma^*\;} & X^* \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi^*} \\ X_\infty & \xleftarrow{\;\varsigma\;} & X_\infty^* \end{array}$$

such that there is a T*-invariant residual subset* X_0^* *of* X^* *having the following property: for all* $x \in X_0^*$, *for any non-empty open subsets* $V_1, \ldots, V_d$ *of* X^* *with* $\pi(x) \in \bigcap_{i=1}^d \pi^*(V_i)$ *and essentially distinct non-constant integral polynomials* $p_1, p_2, \ldots, p_d$ *with* $p_i(0) = 0$, $i = 1, 2, \ldots, d$, *there is some* $n \in \mathbb{N}$ *such that*

$$x \in T^{-p_1(n)} V_1 \cap T^{-p_2(n)} V_2 \cap \ldots \cap T^{-p_d(n)} V_d.$$

Reducing to Pro-nilsystems

Theorem

Let (X,T) be a minimal pro-nilsystem. Let $\mathscr{A} = \{p_1, p_2, \ldots, p_d\}$ be a family of non-constant essentially distinct integral polynomialswith $p_1(0) = \cdots = p_d(0) = 0$. Then we have

1. *The system $(N_\infty(X,\mathscr{A}), \langle T^\infty, \sigma\rangle)$ is a minimal pro-nilsystem.*
2. *For each $x \in X$, the system $(\overline{\mathscr{O}}(\omega_x^{\mathscr{A}}, \sigma), \sigma)$ is a minimal pro-nilsystem.*

Theorem (H.-Shao-Ye, 2022)

Let (X,T) be a minimal t.d.s. and let $\mathscr{A} = \{p_1, p_2, \cdots, p_d\}$ be an family of integral polynomials satisfying $(\spadesuit)$. Then $(N_\infty(X,\mathscr{A}), \langle T^\infty, \sigma\rangle)$ is an M-system.

Thank You!

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