The weak and strong elliptic Harnack inequalities

Jiaxin Hu (Tsinghua University)

(Joint with Zhenyu Yu)

December 11-15, 2023 (Hong Kong)

The weak and strong elliptic Harnack inequalities

Basically, there are two different types of elliptic Harnack inequality. One is the weak Harnack inequality, and the other is the strong Harnack inequality. It turns out that they are equivalent to each other when the Dirichlet form is strongly local, where both the jump measure and the killing measure vanish.

However, the situation is quite different for the non-local DF, where the jump measure does not vanish. In fact, they cannot be equivalent in general. Of course, we always have

$$\text{strong} \Rightarrow \text{weak},$$

but the converse is not true.

- **Main result:** For any regular resurrected Dirichlet form, if (VD), (RVD) hold, then

$$(\mathrm{UE}) + (\mathrm{LLE}) \Leftrightarrow (\mathrm{wEH}) + (\mathrm{E}) + (\mathrm{J}_{\leq}).$$

If further (UJS) holds, then

$$(\mathrm{UE}) + (\mathrm{LLE}) \Leftrightarrow (\mathrm{sEH}) + (\mathrm{E}) + (\mathrm{J}_{\leq}).$$

(We will state what are these conditions.)

Our results: if condition (UJS) holds, then

$$\begin{aligned}(\mathrm{UE})+(\mathrm{LLE}) &\Leftrightarrow (\mathrm{wEH})+(\mathrm{E})+(\mathrm{J}_{\leq}) \\ &\Leftrightarrow (\mathrm{sEH})+(\mathrm{E})+(\mathrm{J}_{\leq}).\end{aligned}$$

- If the DF is strongly local ($J \equiv 0$), the above equivalences become

$$\begin{aligned}(\mathrm{UE})+(\mathrm{LLE}) &\Leftrightarrow (\mathrm{wEH})+(\mathrm{E}) \\ &\Leftrightarrow (\mathrm{sEH})+(\mathrm{E}).\end{aligned}$$

This conclusion was already obtained by Grigor'yan-Hu-Lau (2015, from Corollary 7.3 to Theorem 7.8). In fact, we obtained

weak Harnack $(\mathrm{wEH}) \Leftrightarrow (\mathrm{sEH})$ strong Harnack

for a strongly local DF when the measure satisfies the doubling condition, nothing else is required.

Classical elliptic Harnack inequality

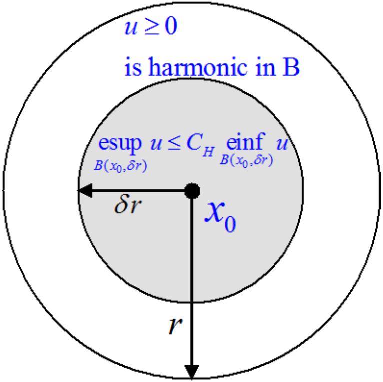
Our motivation

In 1961, Moser proved that, for any non-negative, **harmonic** function u in any ball B (with respect to the symmetric, uniformly elliptic divergence-form operator, which corresponds to a strongly local Dirichlet form in $L^2(dx)$)

$$\sup_{\frac{1}{2}B} u \leq C \inf_{\frac{1}{2}B} u. \tag{1}$$

The importance is that the constant $C \geq 1$ here is **universal** in the sense that it is independent not only of function u, but also of ball B. This inequality says that any non-negative harmonic function in a ball is **nearly constant** around the center.

Nowadays, this inequality is called the elliptic Harnack inequality.



The oscillation of harmonic function around the center is small.

Since the celebrated work by Moser, there has been a lot of work devoted in this direction. Here we are only concerned with **equivalence** conditions for the heat kernel estimate involving **Harnack inequality**.

Heat kernel and Harnack inequality

In this direction,

- Grigor'yan-Hu (2014, Canadian JM) proved the following equivalence: for a strongly local Dirichlet form

$$(\mathrm{UE}_{\mathrm{exp}}) + (\mathrm{NLE}) \Leftrightarrow (\mathrm{H}) + (\mathrm{E}).$$

if the measure is α-regular.

★ **condition** $(\mathrm{UE}_{\mathrm{exp}})$: $p_t(x,y) \leq \frac{C}{t^{\alpha/\beta}} \exp\left(-c\left(\frac{d(x,y)}{t^{1/\beta}}\right)^{\beta/(\beta-1)}\right)$.

★ **condition** (NLE): $p_t(x,y) \geq \frac{c}{t^{\alpha/\beta}}$ whenever $d(x,y) \leq \varepsilon t^{1/\beta}$.

★ **condition** (E): $\|G^B\mathbf{1}\|_{L^\infty} \leq Cr^\beta$ and $\operatorname{einf}_{\delta B} G^B\mathbf{1} \geq C^{-1}r^\beta$ for any ball B of radius r, where $G^B\mathbf{1} = \int_0^\infty P_t 1\, dt$ is the **mean exit time** from ball B.

Question: What happens when the DF is **non-local**?

This is the very issue we are going to address in this talk. It can be imagined that the above classical Harnack inequality

$$\sup_{\frac{1}{2}B} u \leq C \inf_{\frac{1}{2}B} u$$

no longer holds for non-local DF.

- What are **appropriate versions** of the Harnack inequality?
- How does the **jump** part (measure) play a role in the Harnack inequality?

The **weak elliptic Harnack** inequality:

$$\left(\fint_{B_r} u^p d\mu\right)^{1/p} \leq C\left(\operatorname*{einf}_{B_r} u + w(B_r) T_{\frac{3}{4}B_R, B_R}(u_-)\right)$$

for any two concentric balls $B_r \subset B_R$ and any non-negative harmonic

return step2 function u in B_R, where

$$T_{\frac{3}{4}B_R, B_R}(u_-) := \operatorname*{esup}_{x \in \frac{3}{4}B_R} \int_{M \setminus B_R} u_-(y) J(x,y) d\mu(y)$$

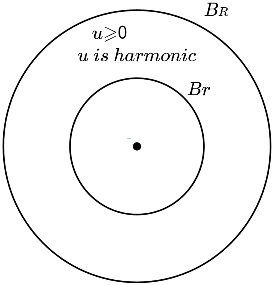
for a constant $0 < p < 1$ and

$$0 < R < \sigma \overline{R}, \quad 0 < r \leq \delta R,$$

with a localized parameter $\overline{R} \in (0, \infty]$ (possibly $\overline{R} = \infty$).

Weak elliptic Harnack inequality (continued)

The weak Harnack inequality says that the *average* of a powered function u^p of u for some $p \in (0, 1)$ over a smaller ball B_r can be controlled by its *infimum* $+$ the *product* of the scaling funciton with *a tail* of the negative part u_- of function u outside a bigger ball B_R.



$$\left(\frac{1}{\mu(B_r)}\int_{B_r} u^p \, d\mu\right)^{1/p} \le C\left\{\inf_{B_r} u + W(B_r)T_{B_{\frac{3}{4}R}, B_R}(u_-)\right\}$$

The weak elliptic Harnack inequality

$$\left(\fint_{B_r} u^p d\mu \right)^{1/p} \leq C \left(\operatorname*{einf}_{B_r} u + w(B_r) T_{\frac{3}{4}B_R, B_R}(u_-) \right). \tag{2}$$

◂ return seh

- If the Dirichlet form is *strongly local*, then the weak elliptic Harnack inequality becomes simpler

$$\left(\fint_{B_r} u^p d\mu \right)^{1/p} \leq C \operatorname*{einf}_{B_r} u \quad \text{(weak Harnack)}$$

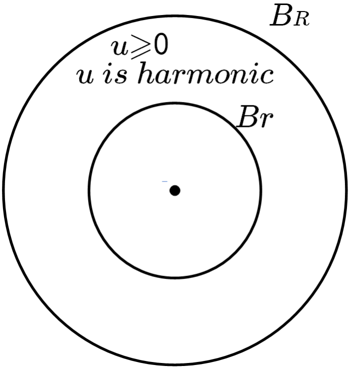
since $T_{\frac{3}{4}B_R, B_R}(u_-) \equiv 0$.

Strong elliptic Harnack inequality

The **strong elliptic Harnack** inequality

$$\operatorname*{esup}_{B_r} u \leq C\left(\operatorname*{einf}_{B_r} u + w(B_r)\, T_{\frac{3}{4}B_R, B_R}(u_-)\right), \qquad \text{(sEH)}$$

that is, the left term in (2) ▸ wEH is replaced by the supremum.



$$\operatorname{esup}_{B_r} u \leq C\left\{\operatorname*{einf}_{B_r} u + W(B_r)\, T_{B_{\frac{3}{4}R}, B_R}(u_-)\right\}$$

Clearly,

$$(\mathrm{sEH}) \Rightarrow (\mathrm{wEH}),$$

since it is clear that

$$\left(\fint_{B_r} u^p d\mu \right)^{1/p} \leq \operatorname*{esup}_{B_r} u.$$

The converse is not true, even the basic **conditions** (PI)**,** (TJ) **and** (E) (thus **condition** (Gcap) also holds), see Example 5.4 by **Chen-Kumagai-Wang** (2020, JEMS).

Framework (doubling space)

Our framework.

- (M, d, μ): a doubling space.

that is, (M, d) is a locally compact, separable metric space, and μ is a Radon measure (locally finite, inner regular) with full support ($\mu(\Omega) > 0$ for any open $\Omega \neq \emptyset$), which satisfies the doubling condition :

$$V(x, 2r) \leq CV(x, r) \text{ for all } x \text{ in } M \text{ and } r > 0,$$

where $V(x, r) := \mu(B(x, r))$ and $B(x, r)$ is an open metric ball.

The fractal is the desired model of the metric space in mind.

Framework (reverse volume doubling condition)

The measure μ also satisfies the following condition.

- The *reverse volume doubling condition* (RVD): There exist two positive constants $C \geq 1$ and d_1 such that for all $x \in M$ and $0 < r \leq R < \overline{R}$

$$\frac{V(x,R)}{V(x,r)} \geq C^{-1}\left(\frac{R}{r}\right)^{d_1}.$$

Known: if (M,d) is *connected* and *unbounded*, then

$$(VD) \Rightarrow (RVD),$$

see for example Grigor'yan-Hu (2014, Moscow JM).

Framework (the scaling function)

- The **scaling function** $w(x, r)$. The function $w(x, \cdot)$ is *continuous, strictly increasing*, $w(x, 0) = 0$, for any fixed x in M, and there exist positive constants C_1, C_2 and $\beta_2 \geq \beta_1$ such that for all $0 < r \leq R < \infty$ and all $x, y \in M$ with $d(x, y) \leq R$,

$$C_1 \left(\frac{R}{r}\right)^{\beta_1} \leq \frac{w(x, R)}{w(y, r)} \leq C_2 \left(\frac{R}{r}\right)^{\beta_2}.$$

- For example, $w(x, r) = r^\beta$ for $\beta > 0$.

Framework (regular resurrected DF)

- $(\mathcal{E}, \mathcal{F})$: a **regular resurrected DF** in $L^2(M, \mu)$:

$$\mathcal{E}(u,u) = \mathcal{E}^{(L)}(u,u) + \iint_{M\times M} (u(x) - u(y))^2 dj(x,y).$$

  The name '**resurrected**' was introduced by Fukushima-Oshima-Takeda on p. 186 in the book (2011).

- In this talk, note that the **jump kernel** J always exists so that

$$dj(x,y) = J(x,y)d\mu(x)d\mu(y).$$

  (If $dj \equiv 0$ then $\mathcal{E} = \mathcal{E}^{(L)}$ is strongly local).

Condition (Gcap)

Generalized capacity condition (Gcap):

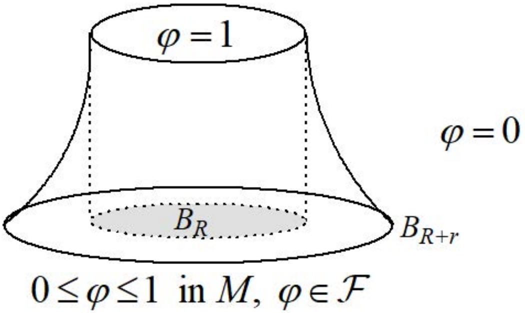
- There exists a constant $C > 0$ such that, for any $u \in \mathcal{F}' \cap L^\infty$ (where $\mathcal{F}' = \mathcal{F} + \text{const}$) and for any two concentric balls B_R, B_{R+r},

$$\mathcal{E}\left(u^2\varphi, \varphi\right) \leq C \sup_{x \in B_{R+r}} \frac{1}{w(x,r)} \int_{B_{R+r}} u^2 d\mu,$$

where $\varphi \in \text{cutoff}\left(B_R, B_{R+r}\right)$.

Note that here C is **universal** that is **independent** of balls B_R, B_{R+r} and functions u, φ.

The cutoff function



- $(\mathrm{Gcap}) \Longrightarrow (\mathrm{Cap}_{\leq})$ (the capacity for two concentric balls**).**
  Recall that the **capacity** $\mathrm{cap}(A, \Omega)$ is defined by

$$\mathrm{cap}(A,\Omega) := \inf\left\{\mathcal{E}(\varphi,\varphi) : \varphi \text{ is a cutoff function of } (A,\Omega)\right\}.$$

- For example, if $w(x,r) = r^{\beta}$, then (Gcap) for $u = 1$ implies that

$$\mathcal{E}(\varphi,\varphi) = \mathcal{E}\left(u^2\varphi,\varphi\right) \leq \frac{C}{r^{\beta}} \int_{B_{R+r}} u^2 d\mu = \frac{C\mu(B_{R+r})}{r^{\beta}}$$

  so that **condition** $(\mathrm{Cap}_{\leq})$ holds:

$$\mathrm{cap}\left(B_R, B_{R+r}\right) \leq \mathcal{E}(\varphi,\varphi) \leq \frac{C\mu(B_{R+r})}{r^{\beta}}.$$

Examples

- Condition (Gcap) is **not** easy to verify!

  However, (Gcap) is not hard to verify in the following two examples.

- Example 1 (Euclidean space). For $0 < \beta < 2$, let

$$\mathcal{E}(u,u) = \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u(x)-u(y))^2}{|x-y|^{n+\beta}} dydx.$$

  Then (Gcap) **holds**, since we can take ϕ to be the bump function

$$\phi(x) := \frac{(R+r-d(x_0,x))_+}{r} \wedge 1$$

  for which,

$$\mathcal{E}(u^2\phi,\phi) \leq \frac{C}{r^\beta} \int_{B_{R+r}} u^2 d\mu.$$

- Example 2 (ultra-metric space). For an ultra-metric space (M, d):

$$d(x, y) \leq \max\{d(x, z), d(z, y)\} \text{ for all } x, y, z \text{ in } M,$$

we let

$$\mathcal{E}(u, u) = \iint_{M\times M} (u(x) - u(y))^2 J(x, dy) d\mu(x)$$

for a **kernel** $J(x, \cdot)$ satisfying condition (TJ):

$$\sup_{x\in M} \int_{B(x,r)^c} J(x, dy) \leq \frac{C}{r^\beta}.$$

Then (Gcap) **holds**, since $\phi = \mathbf{1}_B \in \mathcal{F}$ for any ball B.

The Poincaré inequality (PI):

- There exist two constants $\kappa \geq 1$, $C > 0$ such that for any metric ball $B := B(x_0, r)$ with $0 < r < \overline{R}/\kappa$ and any $u \in \mathcal{F}' \cap L^\infty$,

$$\int_B (u - u_B)^2 d\mu \leq Cw(B) \left\{ \int_{\kappa B} d\Gamma^{(L)}\langle u \rangle + \iint_{(\kappa B)\times(\kappa B)} (u(x) - u(y))^2 dj(x,y) \right\},$$

where u_B is the average of the function u over B

$$u_B = \frac{1}{\mu(B)} \int_B u d\mu =: \fint_B u d\mu$$

Stable-like upper estimate of heat kernel

- **Condition** (UE): if there exists a pointwise defined heat kernel $p_t(x, y)$ on $(0, \infty) \times M \times M$ such that, for any $x, y \in M$ and any $0 < t < w(x, \overline{R}) \wedge w(y, \overline{R})$,

$$p_t(x, y) \leq C \left(\frac{1}{V(x, w^{-1}(x, t))} \wedge \frac{t}{V(x, y) w(x, y)} \right)$$

  with some positive constant C independent of t, x, y, where $V(x, y) := V(x, d(x, y))$ and similarly $w(x, y)$.

Stable-like upper estimate

- For example, if the metric space is unbounded with $\overline{R} = \infty$ and

$$V(x,r) = r^{\alpha} \quad \text{and} \quad w(x,r) = r^{\beta} \quad (0 < \alpha, \beta < \infty),$$

then **condition** (UE) reads

$$p_t(x,y) \lesssim \frac{1}{t^{\alpha/\beta}} \wedge \frac{t}{d(x,y)^{\alpha+\beta}} \asymp \frac{1}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)}$$

for all $t > 0$ and all points x, y in M.

Localized lower estimate (LLE)

- **Condition** (LLE): if the following two properties are satisfied:
  1. for any bounded open set $\Omega \subset M$, the **Dirichlet heat kernel** $p_t^{\Omega}(x, y)$ **exists**;
  2. there exist $C > 0$ and $\varepsilon \in (0, 1)$ such that, for any ball $B := B(x_0, R)$ with $R \in (0, \overline{R})$ and any $0 < t \le w(x_0, \varepsilon R)$,

$$p_t^B(x, y) \ge \frac{C^{-1}}{V(x_0, w^{-1}(x_0, t))}$$

for μ-almost all x, y in $B(x_0, \varepsilon w^{-1}(x_0, t))$.

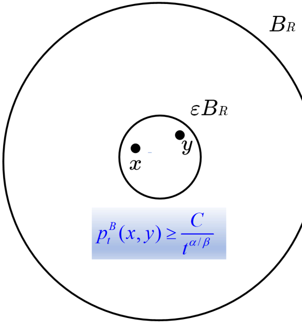
- For example, if

$$V(x,r) = r^{\alpha} \quad \text{and} \quad w(x,r) = r^{\beta} \quad (0 < \alpha, \beta < \infty),$$

then **condition** (LLE) reads

$$p_t^B(x,y) \gtrsim \frac{1}{t^{\alpha/\beta}}$$

for μ-almost all points x, y in εB_R, whenever $0 < t < \varepsilon R^{\beta}$.



Mean exit time estimate

- **Condition** (E): if there exist two constants $C \geq 1$ and $\sigma \in (0,1)$ such that for any ball $B \subset M$ with radius less than $\sigma \overline{R}$,

$$\begin{aligned} \|G^B \mathbf{1}\|_{L^\infty} &\leq Cw(B), \\ \operatorname*{einf}_{\delta B} G^B \mathbf{1} &\geq C^{-1} w(B) \end{aligned}$$

where $G^B \mathbf{1} := \int_0^\infty P_t^B 1 dt$, which satisfies that

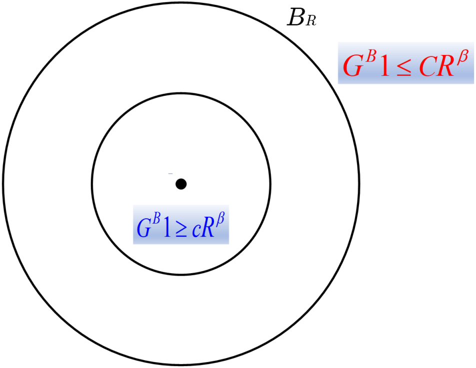
$$\mathcal{E}(G^B \mathbf{1}, \varphi) = \int_B \varphi \, d\mu \ \text{ for any } \varphi \in \mathcal{F}(B).$$

Here $\{P_t^B\}_{t>0}$ is the **heat semigroup** associated with $(\mathcal{E}, \mathcal{F}(B))$.

- For example, if

$$w(x,r) = r^\beta \quad (0 < \beta < \infty),$$

then **condition** (E) is indicated in the following picture



Upper bound of jump kernel

- **Condition** $(\mathrm{J}_{\leq})$: if the *jump kernel* $J(x, y)$ exists on $M \times M$, and there exists a positive constant C such that for $\mu \times \mu$-almost all points $(x, y) \in M \times M \setminus \mathrm{diag}$,

$$J(x, y) \leq \frac{C}{V(x, y)w(x, y)}.$$

Recall that

$$V(x, y) := V(x, d(x, y)) \quad \text{and} \quad w(x, y) := w(x, d(x, y)).$$

Theorem (Yu-H, 2023)

Let $(\mathcal{E}, \mathcal{F})$ *be a regular resurrected Dirichlet form in* L^2 *and let* $\overline{R} = \mathrm{diam}(M)$. *If conditions* (VD), (RVD) *hold, then*

$$(\mathrm{UE}) + (\mathrm{LLE}) \Leftrightarrow (\mathrm{wEH}) + (\mathrm{E}) + (\mathrm{J}_{\leq}).$$

We use the fact that

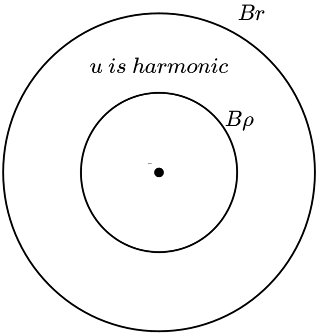
- $(\mathrm{UE}) + (\mathrm{LLE}) \Leftrightarrow (\mathrm{PI}) + (\mathrm{Gcap}) + (\mathrm{J}_{\leq})$ by Grigor'yan-Hu-H (Theorem 2.20, 2023, to appear in Lau's volume).

The **key** argument

- $(\mathrm{PI}) + (\mathrm{Gcap}) + (\mathrm{TJ}) \Rightarrow (\mathrm{wEH})$ by H-Yu (Theorem 1.8, 2023, to appear in Lau's volume). A similar conclusion was obtained by Chen-Kumagai-Wang (2019) under the stronger assumptions.

We will use *condition* (OSL), meaning the *oscillation lemma* for *harmonic* function, which says that there exist three constants θ, σ in $(0, 1)$ and $C > 0$ such that, for any $x_0 \in M$, $0 < r < \sigma\overline{R}$ and any harmonic function u in $B(x_0, r)$,

$$\operatorname*{eosc}_{B(x_0,\rho)} u := \operatorname*{esup}_{B(x_0,\rho)} u - \operatorname*{einf}_{B(x_0,\rho)} u \leq C\left(\frac{\rho}{r}\right)^{\theta} \|u\|_{L^\infty}, \quad 0 < \rho \leq r.$$



Sketch of proof.

- *Step 1.* $(\mathrm{wEH})+(\mathrm{E})+(\mathrm{J}_{\leq})\Rightarrow(\mathrm{UE})+(\mathrm{LLE})$.

  In fact, we can show

$$(\mathrm{wEH})+(\mathrm{TJ})\Rightarrow(\mathrm{OSL}),$$

and then use the known assertion

$$(\mathrm{OSL})+(\mathrm{E})+(\mathrm{TJ}_2)\Rightarrow(\mathrm{LLE})$$

by Grigor'yan-Hu-H (2022, TP-Gcap), see also Chen-Kumagai-Wang (2020, JEMS) under the slightly stronger assumptions.

For upper estimate, by Grigor'yan-Hu-H (2023, TP-LE),

$$\begin{aligned}(\mathrm{LLE}) &\Rightarrow (\mathrm{PI})+(\mathrm{Gcap}),\\ (\mathrm{PI})+(\mathrm{Gcap})+(\mathrm{J}_{\leq}) &\Rightarrow (\mathrm{UE}).\end{aligned}$$

- *Step 2.* (UE) + (LLE) $\Rightarrow$ (wEH) + (E) + ($J_{\leq}$).

  In fact, by Grigor'yan-Hu-H (2023, TP-LE),

$$(\mathrm{UE}) + (\mathrm{LLE}) \Leftrightarrow (\mathrm{PI}) + (\mathrm{Gcap}) + (\mathrm{J}_{\leq}),$$

and use the facts that

$$\begin{aligned} (\mathrm{PI}) &\Rightarrow (\mathrm{FK}) \Rightarrow (\mathrm{E}_{\leq}), \\ (\mathrm{Gcap}) &\Rightarrow (\mathrm{Cap}_{\leq}). \end{aligned}$$

Then, by H-Yu (2023),

$$\begin{aligned} (\mathrm{Gcap}) + (\mathrm{TJ}) + (\mathrm{PI}) &\Rightarrow (\mathrm{wEH}_{+}) \text{ goto } \Rightarrow (\mathrm{wEH}), \\ (\mathrm{Cap}_{\leq}) + (\mathrm{wEH}_{+}) &\Rightarrow (\mathrm{E}_{\geq}), \end{aligned}$$

where the sign "$+$" means for **superharmonic** function (instead of harmonic).

Condition (UJS)

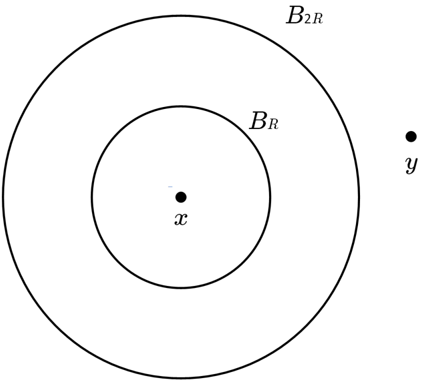
We further look at the equivalence condition for the heat kernel estimate involving the strong elliptic Harnack inequality. To do this, we need condition (UJS).

- **condition** (UJS) (meaning the *upper jumping smoothness* for the jump kernel): if the jump kernel $J(x, y)$ exists on $M \times M$ and satisfies that, for any $0 < R < \sigma \overline{R}$ and for $\mu \times \mu$-almost all points (x, y) in $M \times M$ with $d(x, y) \geq 2R$,

$$J(x, y) \leq C \fint_{B(x,R)} J(z, y) d\mu(z)$$

with some two positive constants C and $\sigma \in (0, 1)$ independent of x, y, R.

Condition (UJS) says that the value of the function $J(\cdot, y)$ at a point x is bounded from above by its average over a ball around this point, which reflects a certain degree of homogeneity of the function $J(\cdot, y)$.



$$J(x,y) \leq C \frac{1}{\mu(B(x,R))} \int_{B(x,R)} J(z,y)\, d\mu(z)$$

This condition was introduced by **Barlow-Bass-Kumagai** (2009, Math Z.).

Examples

- $J \equiv 0$ (**strongly local DF**). Condition (UJS) is trivially satisfied.
- Condition (UJS) is satisfied for

$$J(x,y) = \frac{1}{|x-y|^{n+\beta}}$$

in $\mathbb{R}^n$, since for any point y with $|y-x| > 2R$ and any point z in $B(x,R)$,

$$|z-y| \leq |z-x| + |x-y| \leq R + |x-y| \leq \frac{3}{2}|x-y|,$$

which gives that

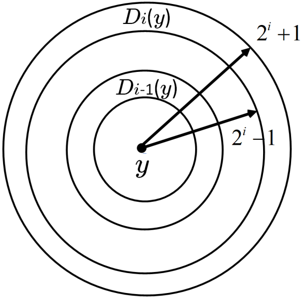
$$\begin{aligned} \fint_{B(x,R)} J(z,y)dz &= \fint_{B(x,R)} \frac{1}{|z-y|^{n+\beta}} dz \\ &\geq \left(\frac{3}{2}|x-y|\right)^{-(n+\beta)} = \left(\frac{3}{2}\right)^{-(n+\beta)} J(x,y). \end{aligned}$$

- Condition (UJS) **fails**. Let $J(x,y)$ be defined in $\mathbb{R}^n$ by

$$J(x,y) := \frac{1}{|x-y|^{n+\beta}} + \frac{\mathbf{1}_D(x,y)}{|x-y|^{n+\theta}} \tag{3}$$

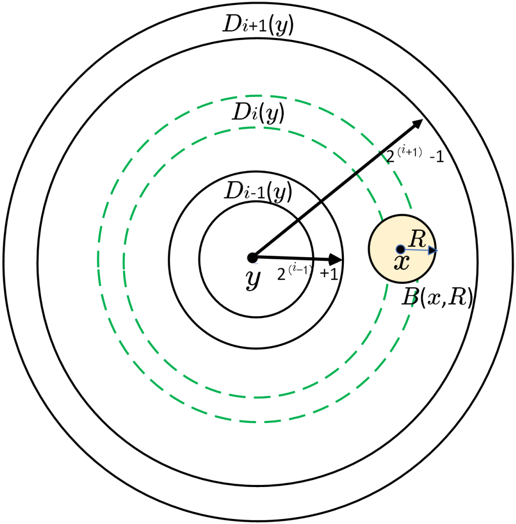
where $1 < \beta < 2$ and $1 \le \theta < \beta$, and the set $D \subset \mathbb{R}^n \times \mathbb{R}^n$ is the disjoint union of $\{D_i\}_{i=1}^{\infty}$: $D := \cup_{i=1}^{\infty} D_i$ with

$$D_i := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : 2^i - 1 < |x-y| < 2^i + 1\}.$$



$$D = \bigcup_{i=1}^{\infty} D_i$$

Condition (UJS) **fails** for any point (x, y) indiacted in the picture, since $J(x,y) \leq C_J \fint_{B(x,R)} J(z,y)dz$ **fails** when $R \in [2^{i-2} - \frac{1}{2}, 2^{i-2} + \frac{1}{2}]$ for large i.



The gap between D_i and D_{i+1} is $2^i - 2$

However, in this example, the following basic conditions

$$(\mathrm{TJ}), (\mathrm{PI}), (\mathrm{Gcap}), (\mathrm{wEH})$$

are all satisfied for a regular Dirichlet form in $L^2(\mathbb{R}^n, dx)$ with the jump kernel $J(x,y)$ defined by (3). **In fact,**

$$\begin{aligned}
(\mathrm{TJ}) &: \int_{B(x,r)^c} J(x,y)dy \le \frac{C}{r^\beta}, \\
(\mathrm{PI}) &: \int_B (u-u_B)^2 dx \le C r^\beta \int_B d\Gamma_B(u), \\
(\mathrm{ABB}) &: \int_\Omega u^2(x) d\Gamma_\Omega(\phi) \le \frac{C}{r^\beta} \int_\Omega u^2(x) dx
\end{aligned}$$

where $\Omega = B(x_0, R')$ is a concentric ball with radius $R' > R + r$, and $(\mathrm{ABB}) \Rightarrow (\mathrm{Gcap})$. Finally (wEH) follows by noting that

$$(\mathrm{Gcap}) + (\mathrm{TJ}) + (\mathrm{PI}) \Rightarrow (\mathrm{wEH}_+) \Rightarrow (\mathrm{wEH})$$

Theorem (Hu-Yu, 2023)

Let $(\mathcal{E}, \mathcal{F})$ *be a regular resurrected Dirichlet form in* L^2 *and let* $\overline{R} = \mathrm{diam}(M)$*. If conditions* (VD), (RVD), (UJS) *are all satisfied, then*

$$\begin{aligned} (\mathrm{UE}) + (\mathrm{LLE}) &\Leftrightarrow (\mathrm{PI}) + (\mathrm{Gcap}) + (\mathrm{J}_{\leq}) \\ &\Leftrightarrow (\mathrm{wEH}) + (\mathrm{E}) + (\mathrm{J}_{\leq}) \\ &\Leftrightarrow (\mathrm{sEH}) + (\mathrm{E}) + (\mathrm{J}_{\leq}). \end{aligned}$$

In particular, if the DF is strongly local, that is, $J \equiv 0$, then

$$\begin{aligned} (\mathrm{UE}) + (\mathrm{LLE}) &\Leftrightarrow (\mathrm{PI}) + (\mathrm{Gcap}) \\ &\Leftrightarrow (\mathrm{wEH}) + (\mathrm{E}) \\ &\Leftrightarrow (\mathrm{sEH}) + (\mathrm{E}), \end{aligned}$$

see Grigor'yan-Hu-Lau (2015).

The **key** argument: if (VD), (Gcap) and (FK) are all satisfied, then

$$(\mathrm{wEH}) + (\mathrm{J}_{\leq}) + (\mathrm{UJS}) \Rightarrow (\mathrm{sEH}) \text{ (H-Yu, 2023).}$$

Consequently, if conditions (VD) and (RVD) are satisfied, then

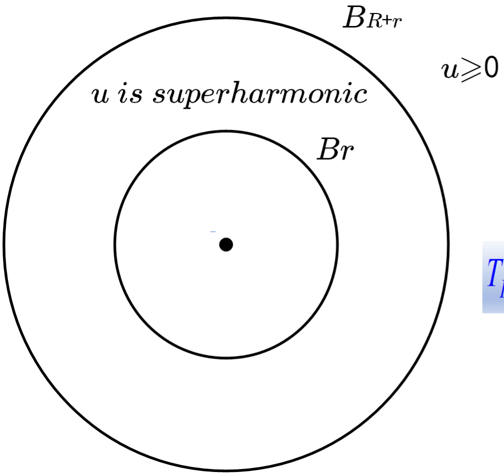
$$(\mathrm{Gcap}) + (\mathrm{PI}) + (\mathrm{J}_{\leq}) + (\mathrm{UJS}) \Rightarrow (\mathrm{sEH}) \text{ (H-Yu, 2023).}$$

Note that condition (UJS) is used to show that

$$T_{B_R, B_{R+r}}(u) = \operatorname*{esup}_{x \in B_R} \int_{B_{R+r}^c} u(y) J(x,y) dy \leq \frac{C}{w(x_0, r)} \left(\frac{R+r}{r} \right)^{c_1} \operatorname*{esup}_{B_{R+r}} u$$

for any **globally non-negative** function $u \in \mathcal{F}' \cap L^\infty$ that is **superharmonic** in B_{R+r} with $\operatorname{esup}_{B_{R+r}} u > 0$.

The importance of this inequality lies in the property that the **supremum** of a superharmonic function u over the *ball* B_{R+r} (instead of over its complement B_{R+r}^c) can control the tail $T_{B_R,B_{R+r}}(u)$.

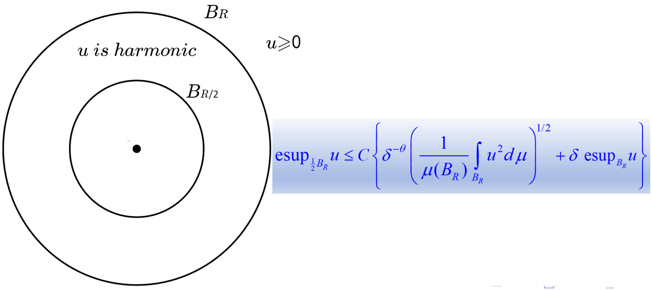


Using the above inequality, we have the following **mean-value inequality (MV):**

$$\operatorname*{esup}_{\frac{1}{2}B_R} u \le C\left[\delta^{-\theta}\left(⨍_{B_R} u^2 d\mu\right)^{1/2} + \delta \operatorname*{esup}_{B_R} u\right]$$

for any $\delta \in (0, 1]$ and any *globally non-negative* function $u \in \mathcal{F}' \cap L^\infty$ that is *harmonic* in any ball $B_R := B(x_0, R)$ with $0 < R < \sigma \overline{R}$.

We emphasize that the constants C, θ, σ are independent of number δ, ball B_R, function u.



After that, we need to show the following implications

$$(\mathrm{MV}) + (\mathrm{wEH}) \Rightarrow (\mathrm{EHI}),$$

$$(\mathrm{FK}) + (\mathrm{J}_{\leq}) + (\mathrm{EHI}) \Rightarrow (\mathrm{sEH}),$$

which finishes the proof.■

- **condition** (EHI): for any **globally non-negative** function $u \in \mathcal{F}' \cap L^{\infty}$ that is harmonic in B_R, we have $\mathrm{esup}_{B_r} u \leq C\, \mathrm{einf}_{B_r} u$ for any $0 < r \leq \delta R < R < \sigma \overline{R}$.