In memory of Professor Kasing LAU

Stationary Random Fields and Trigonometric Multiplicative Chaos

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Outline

Stationary Fields on Compact Abelien Groups

Stationary Random Fields

Data

- G : compact abelian group with Haar measure dx.
- $f_n : G \to \mathbb{C}$: Borel functions
- $\tau_n : G \to G$: endomorphism
- $\{\omega_n\}$: IID random variables with Haar distribution.

Stationary Random Field

$$W(x) \leftrightarrow \{f_n(\tau_n x + \omega_n\} \leftrightarrow \sum_n f_n(\tau_n x + \omega_n).$$

$(W(x_1), \cdots, W(x_r)) \overset{d}{=} (W(x_1 + y), \cdots, W(x_r + y))$.
Examples : $G = \mathbb{T} = \mathbb{R}/\mathbb{Z}$, $\alpha > 0$

$$\sum_{n=1}^{\infty} \frac{f(nx + \omega_n)}{n^\alpha}, \qquad \sum_{n=1}^{\infty} \frac{f(n^2 x + \omega_n)}{n^\alpha}, \qquad \sum_{n=1}^{\infty} \frac{f(2^n x + \omega_n)}{n^\alpha}$$

Problem How about the field

$$W(x) = \sum_n f_n(\tau_n x + \omega_n)?$$

Paley, Zygmund, Wiener, Kahane and others

Paley, Zygmund, Wiener in 1930's

- Rademacher series : $R(x) = \sum \pm a_n e^{2\pi inx}$.
- Steinhaus series : $S(x) = \sum a_n e^{2\pi in(x+\omega_n)}$ our prototype
- Gaussian series : $G(x) = \sum \xi_n a_n e^{2\pi inx}$
- Brownian motion : $W_c(t) = Z_0 t + \sum_{n\neq 0} \frac{Z_n}{2\pi in} e^{2\pi int}$.

Kahane in 1960's Assumption : $Z_n := X_n e^{2\pi i\Phi_n}$ independent and symmetric, $\sum \mathbb{E}|Z_n|^2 < \infty$. The following series defines a nice random function

$$\sum_{n\geq 0} X_n e^{2\pi i\Phi_n} e^{2\pi int} \qquad \text{or} \qquad \sum_{n\geq 0} X_n \cos 2\pi(nt+\Phi_n).$$

Marcus-Pisier (Continuity), Talagrand (Boundedness) $\to$ Banach Geometry.
If $\sum \mathbb{E}|X_n|^2 = \infty$, the series doesn't define a function, neither a measure.
Problem What is the behavior of

$$\sum_{n=1}^{\infty} \frac{\cos 2\pi(nx+\omega_n)}{n^\tau} \quad ? \qquad (\tau \leq \frac{1}{2})$$

The partial sums are multifractal (Fan-Meyer^{2023}).

Example : subtle problem in Dvoretzky random covering

Setting : $d = 1, a_n = 1$ and $f_n(x) = c_n 1_{(0,a/n)}(x)$ with $c_n \geq 0$. We consider the series

$$\sum c_n 1_{(0,a/n)}(x + \omega_n).$$

Theorem (Fan-Kahane[1994])

(1) For any $a > 0$

$$a.s. \forall t \in \mathbb{T}, \quad \sum c_n 1_{(0,a/n)}(x + \omega_n) < \infty \Longleftrightarrow \sum \frac{c_n}{n} < \infty.$$

(2) For $a > 1$

$$a.s. \forall t \in \mathbb{T}, \quad \sum c_n 1_{(0,a/n)}(x + \omega_n) = \infty \Longleftrightarrow \sum \frac{c_n}{n} = \infty.$$

NB. 1 An answer to a question of L. Carleson.

NB. 2 Another answer to the Carleson question (Fan[2001], Barral-Fan[2005]) : $\sum_{n=1}^{N} c_n 1_{(0,a/n)}(x + \omega_n)$ is multifractal .

Convergence of $\sum_{n=1}^\infty \frac{\xi_n}{n^\tau}$ in Hilbert space

Theorem (Fan$^{<2021}$)

Let $\{\xi_n\}_{n\geq 1}$ be IID taking values in a Hilbert space H such that $P(\xi_1 = 0) < 1$. Let $\tau > 0$ and denote $p = \frac{1}{\tau}$.
(1) If the series $\sum_{n=1}^\infty \frac{\xi_n}{n^\tau}$ almost surely converges in H, then $\tau > \frac{1}{2}$.
(2) The series $\sum_{n=1}^\infty \frac{\xi_n}{n^\tau}$ almost surely converges in H if and only if

- (a) $(\text{Case } \tau < 1)$ $\mathbb{E}|\xi|^p < \infty$, $\mathbb{E}\xi = 0$;
- (b) $(\text{Case } \tau = 1)$ $\mathbb{E}|\xi| < \infty$, $\mathbb{E}\xi = 0$, $\sum_{n=1}^\infty n^{-1}\mathbb{E}(\xi 1_{\{|\xi|\leq n\}})$ converges in H;
- (c) $(\text{Case } \tau > 1)$ $\mathbb{E}|\xi|^p < \infty$.

Proof. Consequence of Kolmogorov's three series theorem.

Remarks

- (R1) We assumed that $\{\xi_n\}$ are IID, but no integrability is assumed.
- (R2) What about Banach-valued series ? (Open problem)
- (R3) (Other motivation) Izumi question about $\sum \frac{f(T^n x)}{n}$ (IID answer see (b)).

a.s. almost every convergence of $\sum_{n=1}^{\infty} \frac{f(A_n x+\omega_n)}{n^\tau}$

Assupmtion

- ν : probability measure on G (e.g. Haar measure)
- f : Borel function defined on G (non null function with respect to Haar)
- A_n : endomorphisms (or any map)
- Let $\tau > 0$ and denote $p = \frac{1}{\tau}$.

Theorem (Fan$^{<2021}$)

(1) If $\sum_{n=1}^{\infty} \frac{f(A_n x+\omega_n)}{n^\tau}$ a.s. υ-almost everywhere converges, then $\tau > \frac{1}{2}$.
(2) $\sum_{n=1}^{\infty} \frac{f(A_n x+\omega_n)}{n^\tau}$ almost surely υ-almost everywhere converges iff

(a) (Case $\tau < 1$) $\int |f|^p dm < \infty$, $\int f dm = 0$;

(b) (Case $\tau = 1$) $\int |f| dm < \infty$, $\int f dm = 0$, $\sum_{n=1}^{\infty} n^{-1} \int_{|f|\le n} f dm$ converges;

(c) (Case $\tau > 1$) $\int |f|^p < \infty$.

Proof. Preceeding Thm + Fubini.

a.s. L^p-convergence of $\sum_{n=1}^{\infty} a_n f_n(A_n x + \omega_n)$ on $\mathbb{T}^d$

Assumption

- $2 \leq p < \infty$
- $f_n \in L^p(\mathbb{T}^d) : \int f_n dm = 0, \|f_n\|_p = O(1)$
- A_n : endomorphisms.

Theorem (Fan$^{<2021}$)

The series $\sum_{n=1}^{\infty} a_n f_n(A_n x + \omega_n)$ converges in $L^p(\mathbb{T})$-norm a.s. if there exists an increasing sequence of positive integers $\{N_m\}$ such that

$$\sum_{m=1}^{\infty} \sqrt{\sum_{N_m \leq n < N_{m+1}} a_n^2} < \infty.$$

Remarks

(R1) This condition is stronger than $\sum |a_n|^2 < \infty$

(R2) This condition is satisfied by $a_n = \frac{1}{n^\tau}$ with $\tau > 1/2$.

(R3) Rosenthal inequality is used (Restriction $2 \leq p < \infty$).

Rosenthal inequality

Assume
(1) $p \geq 2$
(2) $\{\xi_k\}$ are independent and centered.
Then for every positive integer n,

$$\mathbb{E}\left|\sum_{k=1}^{n} \xi_k\right|^p \ll \left(\sum_{k=1}^{n} \mathbb{E}|\xi_k|^2\right)^{p/2} + \sum_{k=1}^{n} \mathbb{E}|\xi_k|^p.$$

a.s. uniform convergence of $\sum_{n=1}^\infty a_n f_n(A_n x + \omega_n)$ on $\mathbb{T}^d$

Assumption

- $f_n \in C(\mathbb{T}^d)$: $\int f_n dm = 0, \|f_n\|_\infty = O(1)$. Denote $\Omega(\delta) = \sup_n \Omega(f_n, \delta)$ where $\Omega(f_n, \delta)$ is the modulus of continuity of f_n
- A_n : endomorphisms with polynomial growth

$$\overline{\lim_{n\to\infty}} \frac{\log \|A_n\|}{\log n} < \infty.$$

Theorem (Fan$^{<2021}$)

The series $\sum_{n=1}^\infty a_n f_n(A_n x + \omega_n)$ converges a.s. uniformly if

$$\sum_{m=0}^\infty \Omega(2^{-2^m}) \sum_{2^{2^m} \le n < 2^{2^{m+1}}} |a_n| < \infty, \quad \sum_{m=0}^\infty \sqrt{2^m \sum_{2^{2^m} \le n < 2^{2^{m+1}}} |a_n|^2} < \infty.$$

Remarks

(R1) Good case : $a_n = \frac{1}{n^\tau}$ with $\tau > 1/2$ and $\Omega(\delta) = O(\delta)$.

(R2) Bad case : $\sum \frac{1}{\sqrt{n}} \cos(nx + \omega_n)$–first condition is not satisfied.

Trigonometric Multiplicative Chaos

with Yves Meyer

Random distribution $\sum X_n \cos(n \cdot x + \omega_n)$ on $\mathbb{T}^d$

Assumption

- X_n : independent, subgaussian i.e. $\mathbb{E}e^{\lambda X} \leq e^{a^2\lambda^2/2}$
- $\{X_n\}$ and $\{\omega_n\}$ are indenpendent, $\sum \mathbb{E}|X_n|^2 = \infty$, $\sum \mathbb{E}|X_n|^4 < \infty$.

Partial sums

$$S_N(x) := S_N(x,\omega) := \sum_{0<n\leq N} X_n \cos(n \cdot x + \Phi_n).$$

$\mathbb{E}S_N(t)S_N(s) = H_N(t-s)$, where $H_N(x) = \frac{1}{2}\sum_{0<n\leq N} \mathbb{E}X_n^2 \cos n \cdot x$ and

$$H_N(x) \to H(x) = \frac{1}{2}\sum \mathbb{E}X_n^2 \cos n \cdot x.$$

This is a Function or Distribution of positive type.

For simplicity, we assume in the following

- $d = 1$; $X_n = \rho_n > 0$ deterministic.
- When $\rho_n = \frac{\alpha}{\sqrt{n}}$ $(\alpha > 0)$, we have

$$H_\alpha(t) = \frac{\alpha^2}{2}\sum_{n=1}^{\infty} \frac{\cos nt}{n} = -\frac{\alpha^2}{2}\log\left(2\left|\sin\frac{t}{2}\right|\right)$$

and the $\frac{\alpha^2}{2}$-order Riesz kernel $e^{H_\alpha(t)} \asymp \frac{1}{\left|\sin\frac{t}{2}\right|^{\frac{\alpha^2}{2}}}$.

Trigonometric chaotic measures on $\mathbb{T}$

Trigonometric multiplicative chaos

- The following measure (**trigonometric chaotic measure**) can be defined

$$d\mu_\omega = \prod_{n=1}^{\infty} \exp\left(\rho_n \cos(nt + \omega_n) - \log I_0(\rho_n)\right) dt,$$

- I_0 : the modified Bessel function of first order :

$$I_0(\alpha) = \int_0^{2\pi} e^{\alpha \sin x} \frac{dx}{2\pi} = 1 + \frac{\alpha^2}{4} + O(\alpha^4) \qquad (\alpha \in \mathbb{R}).$$

- The $\mu_\omega \neq 0$ with positive probability if (the energy integral)

$$\int\int e^{H(t-s)} dtds < \infty. \tag{1}$$

- Under (1), we define Peyrière probability measure $\mathcal{Q}$ on $\mathbb{T} \times \Omega$

$$\mathbb{E}_{\mathcal{Q}} h(t,\omega) = \mathbb{E} \int_{\mathbb{T}} h(t,\omega) d\mu_\omega(t)$$

- $\{\cos(nt + \omega_n)\}$ defined on $\mathbb{T} \times \Omega$ are $\mathcal{Q}$-independent

Result : Dimension of chaotic measures

- Assume $\rho_n = \frac{\alpha}{|n|^{d/2}}$.
- More generally, for any finite Borel measure σ on $\mathbb{T}$, we define

$$dQ_\alpha\sigma(t) = \prod_{n=1}^{\infty} \exp\left(\rho_n \cos(n \cdot t + \omega_n) - \log I_0(\rho_n)\right) d\sigma(t)$$

- Denote

$$\tau(d) = \frac{\pi^{d/2}}{\Gamma(d/2)}, \qquad \tau(1) = 1, \tau(2) = \pi, \tau(3) = 2\pi, \tau(4) = \frac{\pi^2}{6}, \cdots .$$

Theorem (Fan-Meyer, 2023)

For any unidimensional measure σ on $\mathbb{T}^d$ we have

(1) If $\dim \sigma < \frac{\alpha^2}{4}\tau(d)$, we have $Q_\alpha\sigma = 0$ a.s. ;

(2) If $\dim \sigma > \frac{\alpha^2}{4}\tau(d)$, we have

$$\dim Q_\alpha\sigma = \dim \sigma - \frac{\alpha^2}{4}\tau(d).$$

Developments in Multiplicative Chaos

- Kolmogorov (1962) : log-normal model of turbulence
- Billard (1965) : martingale method in Dvoretzky covering
- Mandelbrot (1972-1974) : random cascades
- Kahane-Peyrière (1975-1976) : rigourous results for random cascades
- Kahane (1985) : gaussian multiplicative chaos
- Kahane (1987) : general theory of multiplicative chaos
- Fan (1989) : Lévy multiplicative chaos
- Fan (1990) : percolation on trees
- Fan (1990) : a.e. convergence of lacunary series/Riesz product
- Fan-Kahane (1993) : covering numbers in Dvoretzky covering
- Liu, Barral (1990's) : random cascades
- Fan-Kahane (2001) : covering numbers on trees
- Barral-Fan (2002/2005) : covering numbers in Dvoretzky covering
- Kahane-Katznelson (2008) : ergodicity of random integers/Sidon property
- quantum gravity : Barral, Duplantier, Kupiainen, Rhodes, Sheffield, Vargas, ...

General Theory (JP Kahane, Chinese Ann. Math. 1987)

- (T, d) : localemnt compact and metric space
- $(\Omega, \mathcal{A}, P)$: probability space with filtration $\{\mathcal{A}_n\}_{n\geq 1}$
- $P_n(t) := P_n(t,\omega)$ $(n \geq 1)$: processes satisfying

  (a) $P_n(t,\cdot)$ is $\mathcal{A}_n$-mesurable pour tout t ;
  (b) $P_n(\cdot,\omega)$ is Borel measurable for any ω ;
  (c) $P_n(t) \geq 0$, $\mathbb{E}P_n(t) = 1$ for any $t \in T$.

Then we get a T-martingale :

$$\forall t \in T, \quad Q_n(t) := Q_n(t,\omega) = \prod_{j=1}^{n} P_j(t,\omega) \text{ is a martingale.}$$

For positive Radon measure $\sigma \in \mathcal{M}^+(T)$, define a random measure $Q_n\sigma$:

$$Q_n\sigma(A) := \int_A Q_n(t)d\sigma(t) \qquad \big(A \in \mathcal{B}(T)\big)$$

Multiplicative chaos : Fundamental theorem A

Theorem A (JPK 1987)

For any $\sigma \in \mathcal{M}_+(T)$, almost surely $Q_n\sigma$ converges weakly to a random measure, denoted $Q\sigma$.

Proof. $\forall \varphi$, $\int \varphi(t) Q_n(t) d\sigma(t)$ is a convergent martingale. $\square$

Definition. We call $Q\sigma$ a chaotic measure. The chaotic operator $\mathbb{E}Q : \mathcal{M}(T) \to \mathcal{M}(T)$ is defined by

$$\mathbb{E}Q\sigma(A) := \mathbb{E}[Q\sigma(A)].$$

Two extreme cases.

- Q degenerates on σ or σ is Q-irregular if $\mathbb{E}Q\sigma = 0$, i.e. $Q\sigma = 0$, i.e. ;
- Q fully acts on σ or σ is Q-regular if $\mathbb{E}Q\sigma = \sigma$, i.e. $Q_n\sigma(K)$ L^1-converges for any compact set K.

Multiplicative chaos : Fundamental theorem B

Theorem B (JPK 1987)

$\mathbb{E}Q$ is a projection on $\mathcal{M}(T)$. We have the decomposition

$$\mathcal{M}(T) = \mathrm{Im}\mathbb{E}Q \oplus \mathrm{Ker}\mathbb{E}Q$$

Proof. $\forall \sigma$, $\exists$ Borel set B such that

$$\mathbb{E}(Q\sigma(A)|\mathcal{A}_n) = 1_B Q_n \sigma(A).$$

Fundamental problems.

- $\mathrm{Im}\ \mathbb{E}Q = ?$
- $\mathrm{Ker}\mathbb{E}Q = ?$
- $\mathbb{E}[Q\sigma(K)]^h < \infty$?
- $\dim Q\sigma = ?$
- multifractal analysis of $\dim Q\sigma$?

BN. $\mathrm{Im}\ \mathbb{E}Q$ and $\mathrm{Ker}\mathbb{E}Q$ for our trigonometric chaos can be "almost" completely determined.

Percolation on trees

- $\mathcal{T}$: infinite and locally finite tree .
- $T = \partial\mathcal{T}$: compact metric space with metric $d(\xi, \eta) = e^{-|\xi\wedge\eta|}$.
- $\mathbf{p} = (p_n)_{n\geq 1} : 0 < p_n \leq 1$.
- **p**-Bernoulli percolation : Keep (with probability p_n) or Remove (with probability $1 - p_n$) edges in the n–th generation. All actions are independent.
- **Percolation occurs :** if $P(\exists$ an infinite path$) > 0$.
- Problem : When does the percolation occur ?

Theorem (Fan^{1990}, Lyons^{1990})

The **p**–Bernoulli percolation occurs $\Longleftrightarrow \mathrm{Cap}_K\partial\mathcal{T} > 0$ where

$$K(t, s) := K_{\mathbf{p}}(t, s) := \prod_{n=1}^{|t\wedge s|} \frac{1}{p_n}.$$

Percolation on the homogeneous tree $\mathcal{T} = \prod_{n=1}^{\infty} \mathbb{Z}/m_n\mathbb{Z}$

The percolation occurs iff

$$\sum_{n=1}^{\infty} \frac{1}{(p_1 \cdots p_n)(m_1 \cdots m_n)} = \infty.$$

If $m_n = m$ and $p_n = p$, the condition means $p \geq \frac{1}{m}$.

Percolation chaotic operator

- Percolation weights :

$$P_n(\xi) = \frac{1_{\{\text{the } n-\text{th edge of } \xi \text{ is kept}\}}}{p_n}$$

- Percolation chaotic operator : $\mathbb{E}Q_{\mathbf{p}}$.

Theorem (Fan, Thesis (1989))

1. $\mu \in \mathrm{Im}(\mathbb{E}Q_{\mathbf{p}})$ iff $\mu = \sum \mu_k$ with

$$\int\int K(t,s)d\mu(t)d\mu(s) < \infty.$$

2. $\mu \in \mathrm{Ker}(\mathbb{E}Q_{\mathbf{p}})$ iff

$$\exists B \subset \partial\mathcal{T}, \quad \mu(B^c) = 0, \quad \mathrm{Cap}_K(B) = 0.$$

Stationary distributions on torus

Theorem (Fan-Meyer, 2023)

Let $\alpha \in (-2, 2)$. Almost surely $Q_\alpha\lambda$-almost everywhere we have

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{\cos(nt + \omega_n)}{\sqrt{n}} = \frac{\alpha}{2}. \tag{2}$$

Moreover, for any $\eta > 0$, we have the following large deviation

$$\lim_{N\to\infty} \frac{1}{\log N} \log \mathcal{Q}_\alpha \left\{ (t, \omega) : \frac{1}{\log N} \sum_{n=1}^{N} \frac{\cos(nt + \omega_n)}{\sqrt{n}} \notin \frac{\alpha}{2} + [-\eta, \eta] \right\} = -\eta^2. \tag{3}$$

Remarks

(R1) The partial sums are multifractal and for $\alpha \in (-2, 2)$ we have

$$\dim \left\{ \lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{\cos(nt + \omega_n)}{\sqrt{n}} = \frac{\alpha}{2} \right\} \geq 1 - \frac{\alpha^2}{4}$$

(R2) Is $1 - \frac{\alpha^2}{4}$ the exact dimension ? (Problem to be studied)

Theorem (Fan-Meyer, 2023)

Let $\alpha \in (-2, 2)$. Almost surely $Q_\alpha\lambda$-almost everywhere we have a.s. $Q_\alpha\lambda$-almost everywhere

$$\limsup_{N\to\infty} \frac{\sum_{n=1}^{N} \frac{1}{\sqrt{n}} \cos(nt + \omega_n) - \frac{\alpha}{2} \log N}{\sqrt{\log N \log\log\log N}} = 1. \tag{4}$$

If $|\alpha| < 2$ and if $0 \le r < \frac{1}{2}$, a.s. $Q_\alpha\lambda$-almost everywhere we have

$$\limsup_{N\to\infty} \frac{\sum_{n=1}^{N} \frac{1}{n^r} \cos(nt + \omega_n)}{\sqrt{N^{1-2r} \log\log N}} = \sqrt{1 - 2r}. \tag{5}$$

Remarks

(R1) Here we used a law of iterated logarithm due to R. Wittmann (1985).

(R2) If $\frac{1}{2} < r < 1$, $\Omega(S_r, \delta) = O(\delta^{2r-1}\sqrt{\log 1/\delta})$ (cf. Kahane's SRSF, 1985).

(R3) If $r > 1$, the sum function $S_r(x)$ is of class C^1.

(R4) If $r = 1$, $\sum \frac{\cos(nx+\omega_n)}{n}$ should have both differentiable and non-differentiable points (Problem to be studied).

Remarks (continued)

(R5) (on $\mathbb{T}^d$) For $\frac{1}{2} < r < 1$, $\sum_{n=1}^{\infty} \frac{1}{n^r} \cos(A_n x + \omega_n)$ is not studied.

(R6) What is the relation of the above sum (random field) with gaussian field with covariance

$$R(x,y) = \sum_{n=1}^{\infty} \frac{1}{n^{2r}} \cos A_n(x-y).$$

(R7) When $r \le \frac{1}{2}$, how about the generalized gaussian field with covariance defined above. R is no longer continuous and $R(0,0) = +\infty$.

Remarks (continued)

(R8) Trigonometric chaos on $\mathbb{T}^d$ depends on the Jacobi function

$$G(x) = \sum_{k \in \mathbb{Z} \setminus \{0\}} |k|^{-d} \exp(ik \cdot x)$$

which is C^∞ in $\mathbb{T}^d \setminus \{0\}$, and there exists a bounded function E on $\mathbb{T}^d$ such that

$$G(x) = s_d \log \frac{1}{|x|} + E(x).$$

The partial sums

$$S_m(x) = \sum_{|k| \leq m, k \neq 0} |k|^{-d} \exp(ik \cdot x)$$

satisfies

$$S_m(x) \leq s_d \log \frac{1}{|x|} + \text{Constant}.$$

(R9) G is a generalized function of positive type (Schwartz). Is it of σ-positive type (Kahane) ?

Thank You !