Fractal Percolation on Statistically Self-similar and Self-Affine Sets

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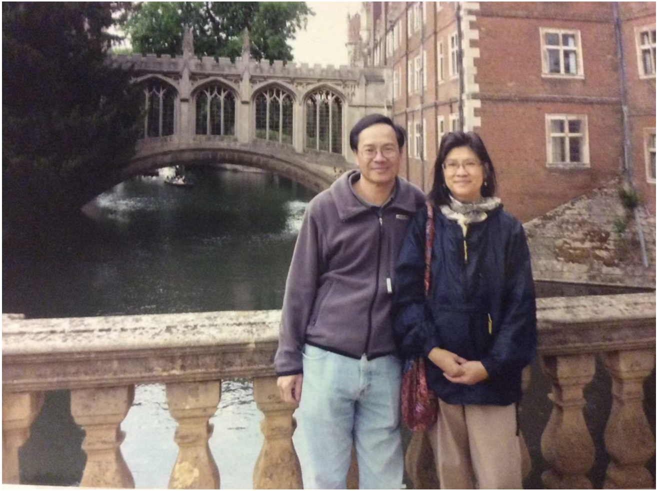


Ka-Sing and Eveline

Overview

• A brief review of statistically self-similar sets, their properties, and percolation on such sets.

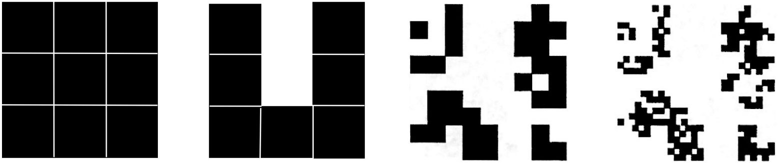
• Statistically self-affine sets, properties, horizontal and vertical crossings, critical probabilities.

• Topological and other properties of statistically self-affine sets.

Joint work with Tianyi Feng (St Andrews)

Construction of statistically self-similar sets

Let $0 < p < 1$ be a probability. Divide the unit square E_0 into 3×3 closed subsquares of side $\frac{1}{3}$ and select each subsquare independently with probability p to get E_1, the union of these selected squares. Repeat with the surviving squares to get subsquares of side $\frac{1}{9}$ to form E_2 and continue in this way to get a decreasing sequence of random sets E_k. Let $F = \cap_{k=0}^{\infty} E_k$.



$E_0 = [0,1]^2$ E_1 E_2 E_3

For an $m \times m$ construction:

- If $p \leq 1/m^2$ then $F = \emptyset$ a.s.; if $p > 1/m^2$ then $\mathbb{P}\{F \neq \emptyset\} > 0$ ($|E_k|$ is a branching process).
- If $p > 1/m^2$ then $\dim_H F = \dim_B F = 2 + \log p / \log m$ a.s. conditional on non-extinction.

Percolation in statistically self-similar sets

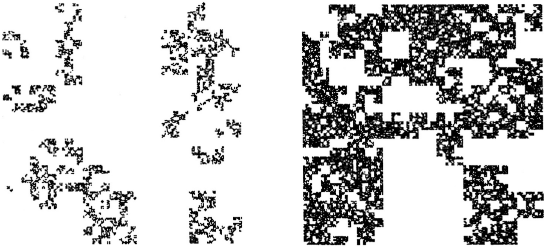
A set F with $p = 0.6$, $p = 0.8$

• Mandelbrot (1974,1977) argued there is a critical probability $0 < p_C < 1$ such F is totally disconnected a.s. if $p < p_C$ but if $p > p_C$ there is positive probability of *percolation* in F i.e. horizontal (or vertical) crossings of $[0,1]^2$ and also of F containing non-trivial connected components.

• Proved by Chayes, Chayes & Durrett (1988), Dekking & Meester (1990).

• Best values known ($m = 3$): $0.784 < p_C < 0.940$ by Don (2015).

Properties of statistically self-similar sets

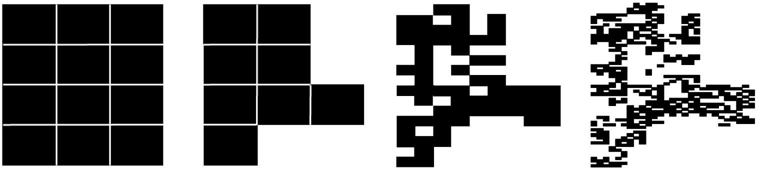


A set F with $p = 0.6$, $p = 0.8$

- Paths in F have large dimension: Chayes (1996), Orzechowski (1998)
- Pure unrectifiability of F: Buczolich, Jäarvenpää[2], Keleti & Pöyhtäri (2021)
- Porosity: Chen, Ojala, Eino & Ville (2017)
- Projections of F: Rams & Simon (2014,2015), Feng & Barral (2018)
- Visible parts of F: Arhosalo, Jäarvenpää[2], Rams & Shmerkin (2012)

Construction of statistically self-affine sets

Let $0 < p < 1$ be a probability. Let $m > n \geq 2$. Divide the unit square E_0 into $m \times n$ closed rectangles of sides $1/n \times 1/m$ and select each rectangle independently with probability p to get E_1 as the union of these rectangles. Repeat with the surviving rectangles to get rectangles of side $1/n^2 \times 1/m^2$ to form E_2; continue to get a decreasing sequence E_k. Let $F = \cap_{k=0}^{\infty} E_k$.



$E_0 = [0,1]^2$ E_1 E_2 E_3

- If $p \leq 1/mn$ then $F = \emptyset$ a.s.; if $p > 1/mn$ then $\mathbb{P}\{F \neq \emptyset\} > 0$ (B-P).
- If $p > 1/mn$ then conditional on non-extinction

$$\dim_H F = \dim_B F = \begin{cases} \log(pnm)/\log n & (1/mn < p \leq 1/m) \\ \log(pm^2)/\log m & (1/m < p \leq 1) \end{cases}$$

(Lalley & Gatzouras (1992), Troscheit (2018), Barral & Feng (2021)).

$p = 0.4$

$p = 0.5$

$p = 0.8$

• There are numbers $0 < p_0$ and $p_1 < 1$ such that F is totally disconnected a.s. if $p < p_0$ and there is positive probability of both horizontal H-crossings and vertical V-crossings of $[0,1]^2$ if $p > p_1$. (Similar proof to self-similar case.)

• Thus there is $0 < p_H < 1$ such that if $0 < p < p_H$ there is a.s no H-crossing and if $p_H < p \leq 1$ then $\mathbb{P}$(there is an H-crossing) > 0.

There is $0 < p_V < 1$ such that if $0 < p < p_V$ there is a.s no V-crossing and if $p_V < p \leq 1$ then $\mathbb{P}$(there is an V-crossing) > 0.

• Question: Is $p_H < p_V$?

$p = 0.4$ $p = 0.5$ $p = 0.8$

Write $\theta_H(p) := \mathbb{P}\{F \text{ contains an H-crossing of } [0,1]^2\}$;
$\theta_V(p) := \mathbb{P}\{F \text{ contains an V-crossing of } [0,1]^2\}$.

Theorem (Tianyi Feng & F) Let $m > n \geq 2$ and $0 < p \leq 1$. Then $\theta_H(p) > 0$ if and only if $\theta_V(p) > 0$. In particular $p_H = p_V$.

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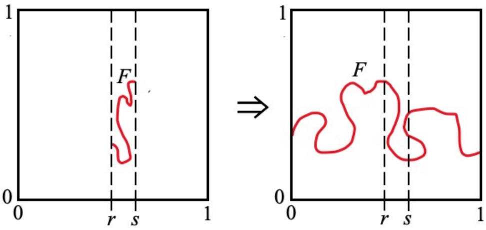
Theorem (Tianyi Feng & F) Let $m > n \geq 2$ and $0 < p \leq 1$. Then $\theta_H(p) > 0$ if and only if $\theta_V(p) > 0$. In particular $p_H = p_V$.

Proposition Let $[r, s] \subset [0, 1]$ be an interval.
If $\mathbb{P}\{F \text{ contains an H-crossing of } [r,s] \times [0,1]\} > 0$ then

$$\theta_H(p) = \mathbb{P}\{F \text{ contains an H-crossing of } [0,1]^2\} > 0.$$



Similarly if $\mathbb{P}\{F \text{ contains a V-crossing of } [0,1] \times [r,s]\} > 0$ then

$$\theta_H(p) = \mathbb{P}\{F \text{ contains a V-crossing of } [0,1]^2\} > 0.$$

Four ideas are needed for the proof

• F is *statistically self-affine*, that is the process of selecting sub-rectangles of each level-k rectangle has the same distribution as the whole process but scaled by a factors $n^{-k} \times m^{-k}$.

• If an event of positive probability is a finite or countable union of sub-events, then at least one of the sub-events has positive probability.

• Let F_k be the set obtained from by selecting *all* rectangles from level-1 to level-$(k-1)$ and starting the random process from the level-k rectangles. Then events such as crossings occur with positive probability in F if and only if it they occur with positive probability in F_k for some, and all, $k \in \mathbb{N}$ (since there is a positive probability of selecting every level-j rectangle for all $1 \leq j \leq k-1$).

• Harris' inequality (or FKG inequality): if A, B are increasing events then $\mathbb{P}(A \cap B) \geq \mathbb{P}(A)\mathbb{P}(B)$. (An event C is *increasing* if C occurs for some selection of rectangles implies that C occurs for every larger (by inclusion) selection of rectangles.)

Proof of Proposition

We adapt the method used by Dekking and Meester to show a phase transition in the self-similar case.

1. Reduction to a LH column. Assume
$\mathbb{P}\{\exists$ an H-crossing of $[r,s]\times[0,1]$ in $F\} > 0$ for some $[r,s]\subset[0,1]$.

Choose integers q, a such that $[an^{-q},(a+1)n^{-q}]\subset[r,s]$. Then

$\mathbb{P}\{\exists$ an H-crossing in F of $[an^{-q},(a+1)n^{-q}]\times[0,1]\} > 0$ so

$\mathbb{P}\{\exists$ an H-crossing in F_q of $[an^{-q},(a+1)n^{-q}]\times[0,1]\} > 0$ so

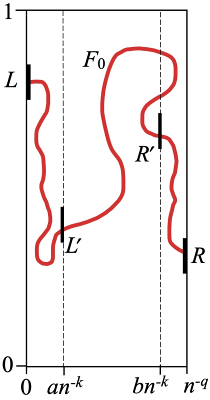
$\mathbb{P}\{\exists$ an H-crossing in F_q of $[0,n^{-q}]\times[0,1]\} > 0$
as $F_q\cap([an^{-q},(a+1)n^{-q}]\times[0,1])$ has the same distribution for all a.

2. No straight line segments.
For $0<p<1$, almost surely the sets F and $F_k (k\geq 1)$ contain no horizontal or vertical line segments.

3. Construction of a linking column.

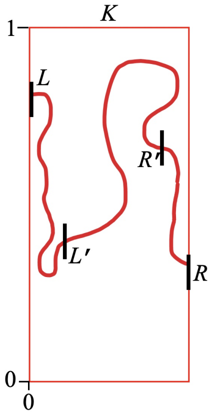
Assume $\mathbb{P}\{\exists$ an H-crossing in F_q of $[0, n^{-q}] \times [0,1]\} > 0$.

Claim: There is $k > q$,integers $0 < a < \frac{1}{2}n^{k-q} < b < n^{k-q}$ and vertical segments L, L', R, R' all of the form $[cm^{-k}, dm^{-k}]$ offset as shown, such that with positive probability p_0, F_k includes a connected component F_0 joining L, L', R, R' such that:
L and L' are connected in $F_k \cap ([0, an^{-k}] \times [0,1])$ and R' and R are connected in $F_k \cap ([b, n^{-q}] \times [0,1])$.
We say that F_k *links* L, L', R', R.

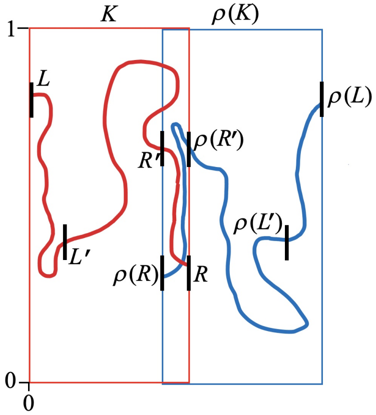


To see this, as a.s. F_k does not contain any horizontal line segment, for each realisation of F_q yielding an H-crossing of $[0, n^{-q}] \times [0,1]$ we can find some k, a, b and L, L', R', R satisfying these conditions. As there are countably many choices for these parameters, we can choose some set of parameters for which satisfies the conditions with positive probability.

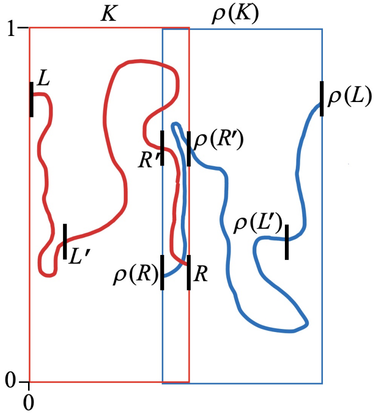
4. Joining the links. Let K be the rectangle $[0, n^{-q}] \times [0, 1]$ with L, L', R', R as in (3). Let ρ denote reflection vertical line mid-way between R' and R.



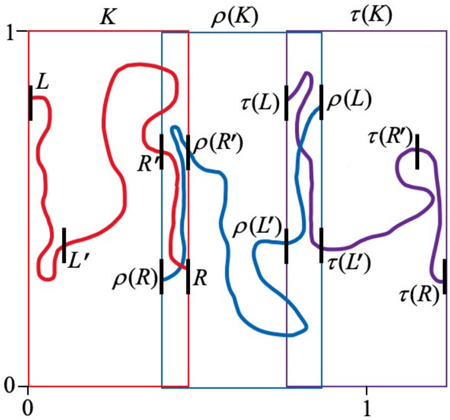
4. Joining the links. Let K be the rectangle $[0, n^{-q}] \times [0, 1]$ with L, L', R', R as in (3). Let ρ denote reflection in the vertical line mid-way between R' and R. If F_k links L, L', R', R and F_k also links $\rho(R), \rho(R'), \rho(L'), \rho(L)$, then F_k connects L to $\rho(L)$.



4. Joining the links. Let K be the rectangle $[0, n^{-q}] \times [0, 1]$ with L, L', R', R as in (3). Let ρ denote reflection in the vertical line mid-way between R and R'. If F_k links L, L', R', R and F_k also links $\rho(R), \rho(R'), \rho(L'), \rho(L)$, then F_k connects L to $\rho(L)$. Both of these are increasing events occurring with probability p_0. By Harris's inequality $\mathbb{P}\{F_k \text{ connects } L \text{ to } \rho(L)\} \geq p_0^2$.



Similarly, we can add a translation τ to link $\tau(L), \tau(L'), \tau(R'), \tau(R)$ to $\rho(R), \rho(R'), \rho(L'), \rho(L)$, also an increasing event with probability p_0. By Harris's inequality $\mathbb{P}\{F_k \text{ connects } L \text{ to } \tau(R)\} \geq p_0^3$.
Continuing in this way, we can eventually build up a sequence of connected components to give an H-crossing of $[0,1]^2$ in F_k, and therefore in F, with positive probability.



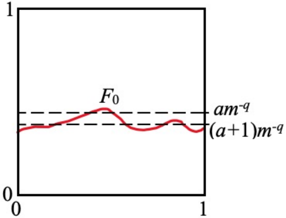
The V-crossing case is similar. □

Theorem Let $m > n \geq 2$ and $0 < p \leq 1$. Then $\theta_H(p) > 0$ if and only if $\theta_V(p) > 0$. In particular $p_H = p_V$.

Proof of Theorem Suppose $\theta_H(p) = \mathbb{P}\{\exists \text{ an H-crossing of } [0,1]^2\} > 0$. Since a.s. F contains no horizontal line segment, we may find q, a such that with positive probability there is a connected component F_0 of F such that

$$\inf\{y : (x,y) \in F_0\} < am^{-q} < (a+1)m^{-q} < \sup\{y : (x,y) \in F_0\},$$

i.e. F_0 includes a V-crossing of $[am^{-q}, (a+1)m^{-q}] \times [0,1]$.



By the Proposition, F_0 includes a V-crossing of $[0,1]^2$ with positive probability, i.e. $\theta_V(p) > 0$.

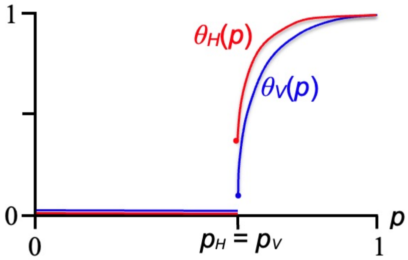
Similarly if $\theta_V(p) > 0$ then $\theta_H(p) > 0$. □

Thus if $0 \leq p < p_H = p_V$ then $\theta_H(p) = \theta_V(p) = 0$
and if $p_H = p_V < p \leq 1$ then $\theta_H(p) > 0$ and $\theta_V(p) > 0$.

We can say more about the functions $\theta_H(p)$ and $\theta_V(p)$.

Theorem Let $m > n \geq 2$. Then $\theta_H(p)$ and $\theta_V(p)$ are right continuous and increasing on [0.1].

Moreover, if $0 \leq p < p_H = p_V$ then $\theta_H(p) = \theta_V(p) = 0$,
and if $p_H = p_V \leq p \leq 1$ then $\theta_H(p) > 0, \theta_V(p) > 0$.



Theorem (i) If $1/mn < p < p_H = p_V$ then conditional on $F \neq \emptyset$ F consists of uncountably many isolated points.

(ii) If $p_H = p_V \leq p < 1$ then conditional on $F \neq \emptyset$ the set F includes infinitely many disjoint non-trivial connected components.

Proof

(i) Let $0 < p < p_H = p_V$. If there is a connected component F_0 of F with distinct $x, y \in F_0$, either a horizontal or vertical strip passes between x and y so is crossed by F_0. By the Proposition there is an H- or V-crossing of $[1, 0]^2$ so $p \geq p_H = p_V$, a contradiction.

(ii) If $p_H = p_V \leq p < 1$ there is a probability $p_\epsilon > 0$ that F contains a non-trivial connected component not touching the boundary of $[0, 1]^2$. Conditioning on the kth-level rectangles in E_k, by self-affinity there is an independent probability $p_\epsilon > 0$ that each rectangle in E_k contains an isolated connected component, so there is an increasingly high probability that at least $\frac{1}{2} p_\epsilon |E_k| \to \infty$ of the rectangles in E_k contain such a component.

Form of statistically self-affine sets

Once we have established that $\theta_H(p) > 0$ if and only if $\theta_V(p) > 0$, various properties follow in a similar way as for statistically self-similar sets. By adapting proofs of Meester (1992):

• Path/arc connectivity: $[0,1]^2$ is crossed both horizontally and vertically by a path/arc in F with positive probability if and only if $p \geq p_H = p_V$ (same value as for topological connectedness).

• Finite number of crossing components: For $p_H = p_V \leq p \leq 1$, almost surely just finitely many disjoint connected components of F cross F horizontally and finitely many cross vertically.

谢谢

Thank you!