On the dimensions of random statistically self-affine Baransky carpets and sponges

(Based on joint works with G. Brunet, and with D.-J. Feng)

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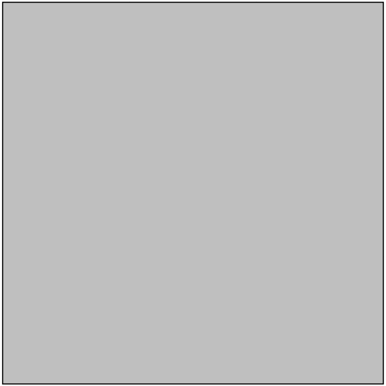
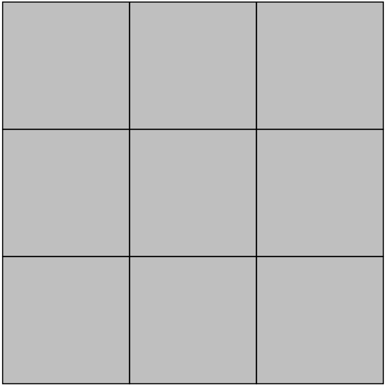
Fractal percolation set

Fix $m \geq 2$. Let $K_0 = [0,1]^2$ be the unit square.



Fix $m \geq 2$. Let $K_0 = [0,1]^2$ be the unit square. Choose a random subcollection $A(\omega)$ of the m^2 subsquares $\{R(i,j) = [im^{-1}, (i+1)m^{-1}] \times [jm^{-1}, (j+1)m^{-1}]\}_{0 \leq i,j \leq m-1}$ of side m^{-1}, according to some given distribution.



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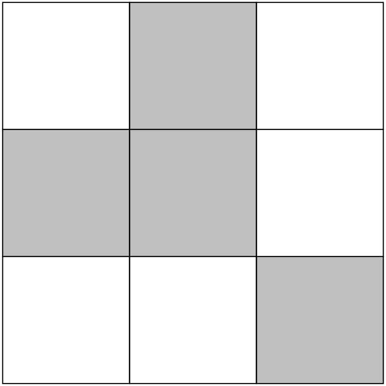
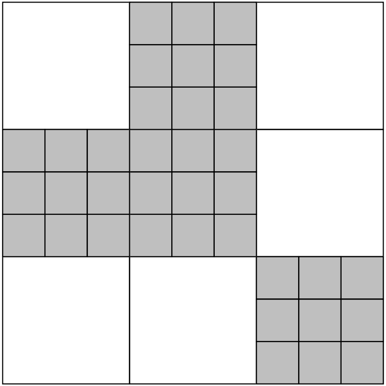
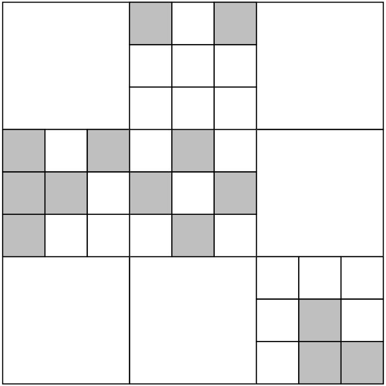
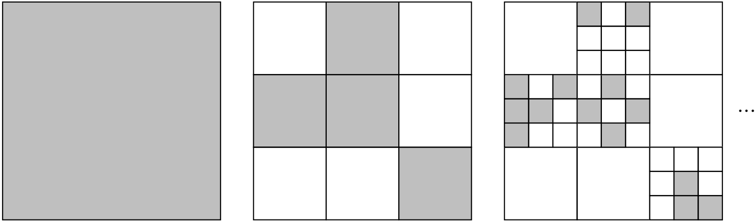


Repeat the selection independently and according to the same law in each selected subsquare.



Repeat the selection independently and according to the same law in each selected subsquare. This yields a set K_2.





K_0 K_1 K_2

$$K = \bigcap_{n \geq 0} K_n.$$

Let $N(\omega) = \#A(\omega)$ denote the (random) number of squares kept at generation 1. One has $K \neq \emptyset$ if and only if $\mathbb{E}(N) > 1$ or $N = 1$ almost surely. In the later case K is a singleton.

Theorem (Falconer (1986))

Let N be the number of surviving squares at the first generation. Suppose $\mathbb{E}\, N > 1$. With probability 1, if $K \neq \emptyset$ then

$$\dim_H K = \dim_B K = \frac{\log(\mathbb{E}\, N)}{\log(m)}.$$

Let N_j be the number of surviving squares in line j, so that $N = \sum_{j=0}^{m-1} N_j$. Suppose $\mathbb{E}\, N > 1$.

Denote by π the orthogonal projection on the vertical axis.

Theorem (Dekking-Grimmett (1988), Falconer (1989))

With probability 1, if $K \neq \emptyset$ then

$$\dim_H \pi K = \dim_B \pi K = \inf_{0 \le s \le 1} \log_m \sum_{i=0}^{m-1} (\mathbb{E}\, N_j)^s.$$

Moreover, $\dim_H \pi K = \dim K$ *iff the infimum is reached at* 1.

Remark: The difficulty of the question partly comes from the fact that it may happen that $0 < \mathbb{E}\, N_j < 1$ for some j.

Before revisiting the previous result, let us mention the result by Rams and Simon. If $\theta \in (-\pi/2, \pi/2)$, denote by π_θ the orthogonal projection on the line $y = \tan(\theta)x$.

Theorem (Rams-Simon (2014, 2015))

Suppose the squares have been chosen independently and with equal probability $p > m^{-2}$. *With probability 1, if* $F \neq \emptyset$, *for all* $\theta \in (-\pi/2, \pi/2)$, *the following holds*

1. $\dim_H \pi_= \min(1, \dim_H K)$;
2. *if* $\dim_H K > 1$ *then* $\pi_\theta K$ *contains an interval.*

Revisiting the two first results with Mandelbrot measures

Take a random non negative vector $W = (W_{i,j})_{0 \le i,j \le m-1}$ such that $\mathbb{E}(\sum_{j=0}^{m-1} W_{i,j}) = 1$.

$W_{0,2}$	$W_{1,2}$	$W_{2,2}$
$W_{0,1}$	$W_{1,1}$	$W_{2,1}$
$W_{0,0}$	$W_{1,0}$	$W_{2,0}$

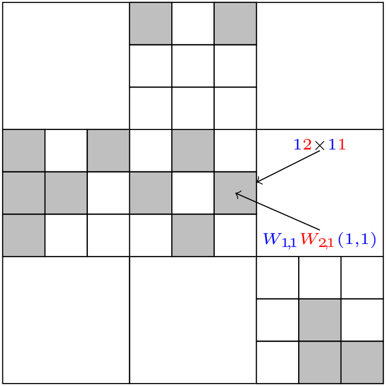
Revisiting the two first results with Mandelbrot measures

Suppose that $\mathbb{E}(N) > 1$. Take a random non negative vector $W = (W_{i,j})_{0\leq i,j\leq m-1}$ such that $\mathbb{E}(\sum_{j=0}^{m-1} W_{i,j}) = 1$. Assume that $W_{i,j} = 0$ if (i,j) does not survive, i.e. $(i,j) \notin A(\omega)$.

0	$W_{1,2}$	0
$W_{0,1}$	$W_{1,1}$	0
0	0	$W_{2,0}$

Set $\mu_1(i_1 \times j_1) = W_{i_1,j_1}$

Next independently in each surviving subsquare $i_1 \times j_1$ take a copie $W(i_1, j_1) = (W_{i_2,j_2}(i_1, j_1))_{0 \le i_2, j_2 \le m-1}$ of W and set



$$\mu_2(i_1 i_2 \times j_1 j_2) = W_{i_1,j_1} W_{i_2,j_2}(i_1, j_1)$$

Iterate: for $n \geq 1$ and $I = i_1 \cdots i_n$ and $J = j_1 \cdots j_n$,

$$\mu_n(I \times J) = W_{i_1,j_1} W_{i_2,j_2}(i_1, j_1) \cdots W_{i_n,j_n}(i_1 \cdots i_{n-1}, j_1 \cdots j_{n-1}),$$

the mass being distributed uniformly.
One has

$$\text{supp}(\mu_n) \subset K_n.$$

Set $\mathcal{A} = \{0, \ldots, m-1\}^2$ and

$$T(\theta) = -\log \mathbb{E} \sum_{(i,j)\in\mathcal{A}} W_{i,j}^{\theta}; \quad \text{note that } T'(1^-) = -\mathbb{E} \sum_{(i,j)\in\mathcal{A}} W_{i,j} \log W_{i,j}.$$

Theorem (Kahane-Peyrière (1976), Kahane (1987))

With probability 1, conditional on $K \neq \emptyset$, the sequence $(\mu_n)_{n\geq 1}$ weakly converges towards a mesure μ supported on K.
If $\mathbb{P}(\#\{(i,j) : W_{i,j} > 0\} = 1) = 1$, then μ is a Dirac mass almost surely.
Otherwise, $\mathbb{P}(\mu \neq 0 | K \neq \emptyset) > 0$ iff $T'(1^-) > 0$, and in this case, conditional on $\mu \neq 0$, then μ is exactly dimensional with $\dim(\mu) = \dim_e(\mu)/\log(m)$, where

$$\dim_e(\mu) = \lim_{n\to\infty} n^{-1} \sum_{|I|=|J|=n} -\mu(I \times J) \log \mu(I \times J) = T'(1^-).$$

Recall that $T(\theta) = -\log \mathbb{E} \sum_{(i,j)\in\mathcal{A}} W_{i,j}^{\theta}$.

Theorem (Falconer-Jin, 2014)

Suppose that $T(\theta) > -\infty$ for some $\theta > 1$ and $T'(1) > 0$. With probability 1, if $\mu \neq 0$, for all θ, the measure $\pi_{\theta *}\mu$ is exact dimentional.

Let

$$\nu = \mathbb{E}(\pi_* \mu).$$

Setting $p_{i,j} = \mathbb{E}(W_{i,j})$, and $q_j = \sum_{i=0}^{m-1} p_{i,j}$ so that $q_0 + q_1 + \ldots + q_{m-1} = 1$, ν is the Bernoulli product measure on $[0,1]$ generated by the probability vector $(q_0, \ldots, q_{m-1})$.

Theorem (B.-Feng, 2018)

Suppose $T'(1^-) > 0$. With probability 1, if $\mu \neq 0$:

1. *If* $\dim(\mu) > \dim(\nu)$, *then* $\pi_*\mu \ll \nu$, *hence* $\dim(\pi_*\mu) = \dim(\nu)$.
2. *If* $\dim(\mu) \leq \dim(\nu)$, *then* $\pi_*\mu \perp \nu$.
   If, moreover, $T(\theta) > -\infty$ *for some* $\theta > 1$, *then* $\pi_*\mu$ *is exact dimensional and* $\dim(\pi_*\mu) = \dim(\mu)$.

Thus, if $T(\theta) > -\infty$ for some $\theta > 1$ and $T'(1) > 0$, if $\mu \neq 0$, then

$$\dim(\pi_*\mu) = \min\{\dim(\mu), \dim(\nu)\}, \quad \text{where } \nu = \mathbb{E}(\pi_*\mu).$$

Ingredients of the proof: The structure of π_μ is as follows.

If $y \in [0,1)$ and $J = J_n(y) = j_1 \cdots j_n$ is the semi-open to the right m-adic interval of generation n containing y, then

$$\pi_*\mu(J) = \sum_{|I|=n} \mu(I \times J) = \nu(J) \cdot Z_J \quad \text{where } Z_J = \sum_{|I|=n} \frac{\mu_n(I \times J)}{\nu(J)} Y(I, J),$$

hence $\pi_*\mu$ is locally essentially the product of its expectation and an inhomogeneous Mandelbrot martingale, more precisely a Mandelbrot martingale in a random environment if one considers $Z_{J_n(y)}$ for ν-almost every y.

To get the dimension of $\pi_*\mu$, one studies its L^q-spectrum and prove that in a neighbourhood of 1,

$$\mathbb{E} \sum_{|J|=n} \pi_*\mu(J)^\theta \leq C_q n \begin{cases} m^{-n \max(\tau_\mu(\theta), \tau_\nu(\theta))} & \text{if } \theta < 1 \\ m^{-n \min(\tau_\mu(\theta), \tau_\nu(\theta))} & \text{if } \theta \geq 1 \end{cases}.$$

This yields

$$\tau'_{\pi_*\mu}(1) = \min(\tau'_\mu(1), \tau'_\nu(1)).$$

Corollary (B.-Feng (2018))

With probability 1, conditionally on $F \neq \emptyset$, one has

$$\begin{aligned}\dim_H \pi(K) &= \dim_B(\pi(K)) \\ &= \inf_{0\le\theta\le 1} \log_m \sum_{j=0}^{m-1} \mathbb{E}(N_j)^\theta \\ &= \max\{\dim_H(\pi_*\mu) : \mu \text{ is a Mandelbrot measure supported on } F\}.\end{aligned}$$

Moreover, the above maximum is not attained at a unique point if and only if the above infimum is attained at $\theta = 0$ and $\sum_{i=0}^{m-1} \log(\mathbb{E}(N_i)) > 0$.

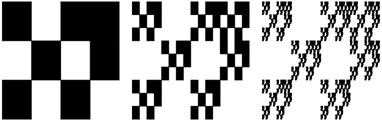
It is also clear that

$$\dim_H K = \sup\{\dim(\mu) : \mu \text{ is a Mandelbrot measure supported on } K\},$$

and the supremum is uniquely attained at the so called "branching measure", that is the Mandelbrot measure associated to $W_{i,j} = \mathbf{1}_{A_\omega}(i,j)/\mathbb{E}(N)$.

Let $m_1 > m_2 \geq 2$ in $\mathbb{N}$, and $\emptyset \neq A \subset \{0,\dots,m_1-1\} \times \{0,\dots,m_2-1\}$. $\#A \geq 2$.
Let K be the attracteur in $\mathbb{T}^2$ of the affine IFS:

$$S = \left\{ f_{i,j} : (x_1,x_2) \in \mathbb{T}^2 \mapsto \left(\frac{x_1+i}{m_1}, \frac{x_2+j}{m_2}\right) : (i,j) \in A \right\}.$$



Let

$$N_j = \#\{i : (i,j) \in A\}, \quad P(\theta) = \log \sum_{j=0}^{m_2-1} N_j^\theta.$$

Theorem (McMullen, Bedford, 1984)

$$\dim_H K = \frac{P(\alpha)}{\log(m_2)} \ \textit{where } \alpha = \frac{\log(m_2)}{\log(m_1)}$$
$$= \max\{\dim(\mu) : \mu \textit{ self-affine supported on } K\} \quad \text{(McMullen)}.$$

If μ_p stands for the self-affine measure on K associated with the probability vector $p = (p_{i,j})_{(i,j)\in A}$, one has the Ledrappier-Young formula

$$\dim(\mu_p) = \frac{1}{\log(m_1)} h_{\mu_p}(T_1) + \Big(\frac{1}{\log(m_2)} - \frac{1}{\log(m_1)}\Big) h_{\pi_{2*}\mu_p}(T_2)$$

where $T_1(x_1, x_2) = (\{m_1 x_1\}, \{m_2 x_2\})$.

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The maximal for $\dim(\mu_p)$ is uniquely attained when

$$p_{i,j} = p_{i,j}^{(\alpha)} = \frac{N_j^\alpha}{\sum_{j'=0}^{m_2-1} N_{j'}^\alpha} \frac{\mathbf{1}_A(i,j)}{N_j}, \quad \alpha = \frac{\log(m_2)}{\log(m_1)};$$

Moreover, $\dim(\mu_{p^{(\alpha)}}) = \dfrac{P(\alpha)}{\log(m_2)}$ and by a simple combinatoric argument $\forall\, x \in K$, $\dim_{\mathrm{loc}}(\mu_{p^{(\alpha)}}, x) \le \dfrac{P(\alpha)}{\log(m_2)}$ (McMullen).

Explanation: for $x=(x_1,x_2)\in K$ and $n\in\mathbb{N}^*$, set $\ell(n)=\left\lceil n\frac{\log(m_1)}{\log(m_2)}\right\rceil\sim\frac{n}{\alpha}$, so that $m_1^n\approx m_2^{\ell(n)}$. Let $I_n(x_1)$ and $J_n(x_2)$ be the m_1-adic and m_2-adic intervals of generation n contening x_1 et x_2 respectively.
Set

$$B_n(x)=I_n(x_1)\times J_{\ell(n)}(x_2)\quad \text{(it is an “almost square” of side } m_2^{-\ell(n)}\text{).}$$

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One has

$$\begin{aligned}\mu(B_n(x)) &= \mu(I_n(x_1)\times J_n(x_2))\cdot \pi_{2*}\mu(J_{\ell(n)-n}(T_2^n(x_2))\\ &= {m_2}^{-\ell(n)\frac{P(\alpha)}{\log(m_2)}}\Big(\prod_{k=1}^{n} N_{x_{2,k}}\Big)^{-1}\Big(\prod_{k=1}^{\ell(n)} N_{x_{2,k}}\Big)^{\alpha}\\ &\approx {m_2}^{-\ell(n)\frac{P(\alpha)}{\log(m_2)}}\cdot\exp(n(u_{\ell(n)}(x_2)-u_n(x_2))).\end{aligned}$$

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Since $(u_n(x_2))_n=\Big(n^{-1}\sum_{k=1}^n\log N_{x_{2,k}}\Big)_n$ is bounded, one has $\limsup_{n\to\infty}u_{\ell(n)}(x_2)-u_n(x_2)\geq 0$, and $\underline{\dim}_{\rm loc}(\mu,x)\leq\frac{P(\alpha)}{\log(m_2)}$.

As we know, McMullen's approach can be generalised to Siepinski sponges in dimension ≥ 3 using more elaborate combinatorics (Kenyon and Peres).

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- To $x_2 \in \pi(K)$ associate $s(x_2) = \limsup_{n\to\infty} \sum_{j=0}^{m_2-1} q_j(x_2,n)\log(N_j)$, where $q_j(x_2,n))$ is the frequency of the digit j among the n first digits of x_2 in basis m_2.
- $\exists$ $(n_k)_{k\in\mathbb{N}}$ such that $q_j(x_2,\ell(n_k))$ converges to q_j for all j and $\sum_{j=0}^{m_2-1} q_j \log(N_j) = s(x_2)$.
- Gathering points of K according to the value of $q = q(x_2)$, discretizing, and letting $(n_k)_{k\in\mathbb{N}}$ vary yields good coverings of K and the inequality

$$\dim_H K \le \sup_q \left\{ \frac{1}{\log(m_2)} h(q) + \frac{1}{\log(m_1)} \sum_{j=0}^{m_2-1} q_j \log(N_j) \right\} = \frac{P(\alpha)}{\log(m_2)}.$$

- Alternative approach, fruitful in higher dimension : write

$$N \sum_{j=0}^{m_2-1} q_j(x_2,N)\log(N_j) + \ell(N) \sum_{j=0}^{m_2-1} -q_j(x_2,\ell(N))\log(q_j(x_2,\ell(N)))$$

$$= N\Big(\sum_{j=0}^{m_2-1} q_j(x_2,N)\log(N_j) + \frac{\ell(N)}{N} \sum_{j=0}^{m_2-1} -q_j(x_2,N)\log(q_j(x_2,N))\Big) + r_N(x_2)$$

with

$$r_N(x_2) = \ell(N)\big(h_{\nu_{q(x_2,\ell(N))}}(T_2) - h_{\nu_{q(x_2,N)}}(T_2)\big),$$

where ν_q is the Bernoulli measure associated with the probab. vector q.

The random case. We come back to the initial fractal percolation model but work on a rectangular grid: $A = A_\omega$ is now a random subset of $\mathcal{A} = \{0, \dots, m_1 - 1\} \times \{0, \dots, m_2 - 1\}$ such that $\mathbb{E}(\#A) > 1$.

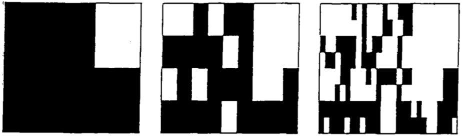

Fig. 1. This shows 3 generations in the construction of a statistically self-affine set. In each generation, every surviving rectangle is divided into 6 congruent rectangles, arranged in 2 rows and 3 columns. Each of these new rectangles is then discarded with probability .3.

(picture from Gatzouras-Lalley's paper).

The result of Gatzouras and Lalley revisited. Let

$$P(\theta) = \log \sum_{j=0}^{m_2-1} \mathbb{E}(N_j)^\theta.$$

and θ_0 be the unique point where P reaches its minimum on $[0,1]$ if P is not constant, and $\theta_0 = 1$ otherwise.

Theorem (Gatzouras-Lalley's (1994); B.-Feng (2021))

With probability 1, if $K \neq \emptyset$,

$$\begin{aligned}\dim_H K &= \frac{P(\alpha)}{\log(m_2)} \text{ where } \alpha = \max\left(\theta_0, \frac{\log(m_2)}{\log(m_1)}\right),\\ &= \max\{\dim_H(\mu) : \mu \text{ is a Mandelbrot measure supported on } K\},\end{aligned}$$

and the maximum is uniquely attained, when

$$W_{i,j} = \mathbf{1}_{\{A_\omega\}}((i,j)) \frac{\mathbb{E}(N_j)^{\theta_0}}{\sum_{j'=0}^{m_2-1} \mathbb{E}(N_j)^{\theta_0}} \frac{1}{\mathbb{E}(N_j)}.$$

Theorem (B.-Feng, 2021)

Let μ be a non degenerate Mandelbrot measure associated with weights W supported on K. Suppose $T(\theta) > -\infty$ for some $\theta > 1$ and $T'(1) > 0$. Set $\nu = \mathbb{E}(\pi_* \mu)$. Then, with probability 1, conditional on $\mu \neq 0$, one has the Ledrappier-Young type formula

$$\dim(\mu) = \tau'_\mu(1) = \frac{1}{\log(m_1)} \dim_e(\mu) + \left(\frac{1}{\log(m_2)} - \frac{1}{\log(m_1)}\right) \dim_e(\pi_* \mu).$$

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If we specify that μ is associated with $W_{i,j} = q_j \mathbf{1}_{A_\omega}((i,j)) \frac{1}{\mathbb{E}(N_j)}$, where q is a probab. vector $(q_j)_{j=0}^{m_2-1}$ such that $q_j > 0$ only if $\mathbb{E}(N_j) > 0$, the associated Mandelbrot measure μ, if not degenerate, satisfies

$$\dim(\mu) = \min\big(d_1(q), d_2(q)\big),$$

where

$$\begin{cases} d_1(q) = \dfrac{h(q)}{\log(m_2)} + \dfrac{\sum_{j=0}^{m_2-1} q_j \log \mathbb{E}(N_j)}{\log(m_1)} \\ d_2(q) = \dfrac{h(q) + \sum_{j=0}^{m_2-1} q_j \log \mathbb{E}(N_j)}{\log(m_2)}. \end{cases}$$

Also, Gatzouras and Lalley show (exploiting Bedford argument) that conditional on $K \neq \emptyset$,

$$\dim_H K \le \sup_q \big\{ \min\big(d_1(q), d_2(q)\big) \big\}.$$

Theorem (B.-Feng, 2021)

If K is statistically self-affine Sierpinski sponge ($d \geq 2$),

$$\dim K = \max\{\dim_H(\mu) : \mu \textit{ is a Mandelbrot measure}\}.$$

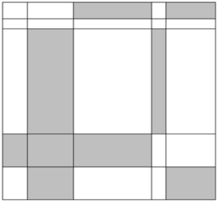
and the maximum is uniquely attained. Also, $\dim K$ *is expressed as the weighted pressure of some potential.*

The dimensions of the Mandelbrot measures is obtained as in the 2-dimensional case.
For the upper bound, our argument combines the alternative Bedford-like approach + combinatoric lemma of Kenyon-Peres to get a nice uncountable family of coverings and upper bounds by weighted pressures of a family of potentials + optimisation over these upper bounds.

Considers in each principal direction $1 \leq k \leq d$ of R^d a linear IFS $\{f_i^{(k)}\}_{i \in A_k}$ made of at least two pairwise distinct affine maps, satisfying the open set condition, and such that $f_i([0,1]) \subset [0,1]$ for each i. For $\boldsymbol{i} = (i_1, \ldots, i_d) \in \prod_{k=1}^d A_k$, set

$$f_{\boldsymbol{i}} = f_{i_1}^{(1)} \otimes \cdots \otimes f_{i_d}^{(d)}.$$

As for the construction of a statistically self-affine Sierpinski carpet or sponge, consider $A_\omega \subset \prod_{k=1}^d A_k$ such that $\mathbb{E}(\#A) > 1$, and perform a percolation according to the distribution of A. The limit set K is called a statistically self-affine Baranski carpet.
If A is deterministic, it is simply the attractor of the IFS $\{f_{\boldsymbol{i}}\}_{\boldsymbol{i} \in \mathcal{A}}$.



(Baranski carpet; picture from Baranski's paper).

Theorem ($d = 2$; Baranski (for the deterministic case, 2008) and Brunet (2022))

With probability 1, conditional on $\{K \neq \emptyset\}$,

$$\dim_H K = \max\left\{\dim_H(\mu) : \mu \textit{ is a Mandelbrot measure supported on } K\right\}.$$

For the random case, the approach is quite similar to that used to revisit the random Sierpinski case using Mandelbrot measures.

However, we note that the fact that Lyapounov exponents are not constant complicates both the computation of the dimensions of Mandelbrot measures (variant of the L^q-spectrum must be used) and the construction of random coverings (but the key observation of Bedford is still very efficient).

To state the following results for $d \geq 3$, we set
$A = \{\boldsymbol{i} \in \prod_{k=1}^{d} A_k : \mathbb{P}(\boldsymbol{i} \in A_\omega) > 0\}$ and consider the coding map

$$\Pi : (x_n)_{n\geq 1} \in A^{\mathbb{N}} \mapsto \lim_{n\to\infty} f_{x_1} \circ \cdots \circ f_{x_n}(0).$$

Let $\mathcal{P}_A = \{p = (p_{\boldsymbol{i}})_{\boldsymbol{i}\in A} \in \mathbb{R}_+^A : \sum_{\boldsymbol{i}\in A} p_{\boldsymbol{i}} = 1\}$.

Given a sequence $\widetilde{p} = (p^{(n)})_{n\geq 1} \in \mathcal{P}_A^{\mathbb{N}}$, denote by $\mu_{\widetilde{p}}$ the image by Π of the inhomogenous Bernoulli product $\otimes_{n=1}^{\infty} \left(\sum_{\boldsymbol{i}\in A} p_{\boldsymbol{i}}^{(n)} \delta_{\boldsymbol{i}}\right)$. The measure $\mu_{\widetilde{p}}$ is supported on K.

The self-affine measures associated with $\{f_{\boldsymbol{i}}\}_{\boldsymbol{i}\in A}$ are the measures $\mu_{\widetilde{p}}$ where $\widetilde{p}$ is constant.

Theorem (Das-Simmons (2017); $d \geq 3$, deterministic case)

$$\begin{aligned}\dim_H K &= \sup\left\{\dim_H(\mu_{\widetilde{p}}) : \widetilde{p} \in \mathcal{P}_A^{\mathbb{N}}\right\} \\ &= \sup\left\{\dim_H(\mu_{\widetilde{p}}) : \widetilde{p} = (r(n))_{n\geq 1}, \begin{cases} r : (0,\infty) \to \mathcal{P}_A^{(0,\infty)} \\ r \textit{ is } C^0 \textit{ and exponentially periodic} \end{cases}\right\}.\end{aligned}$$

Moreover, it happens that
$\dim_H K > \max\{\dim(\mu) : \mu \textit{ is self-affine and supported on } K\}$.

If $\widetilde{p} \in \mathcal{P}_A^{\mathbb{N}}$, define M_p as the inhomogeneous Mandelbrot measure obtained by using at generation n independent copies of $W^{(n)} = (W_{\boldsymbol{i}}^{(n)})_{\boldsymbol{i}\in A}$, where

$$W_{\boldsymbol{i}}^{(n)} = p_{\boldsymbol{i}}^{(n)} \frac{\mathbf{1}_{A_\omega}(\boldsymbol{i})}{\mathbb{P}(\boldsymbol{i} \in A_\omega)}.$$

Theorem (B.-Brunet (2023); $d \geq 3$, random case)

With probability 1, conditional on $K \neq \emptyset$,

$$\begin{aligned}\dim_H K &= \sup\left\{\dim_H(M_{\widetilde{p}}) : \widetilde{p} \in \mathcal{P}_A^{\mathbb{N}}\right\} \\ &= \sup\left\{\dim_H(M_{\widetilde{p}}) : \widetilde{p} = (r(n))_{n\geq 1}, \begin{cases} r : (0,\infty) \to \mathcal{P}_A^{(0,\infty)} \\ r \textit{ is } C^0 \textit{ and exponentially periodic}\end{cases}\right\}.\end{aligned}$$