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Dimension of planar non-conformal attractors with triangular derivative matrices

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joint work with Antti Käenmäki.

Introduction

- Let $\mathcal{S} = \{f_1, \ldots, f_m\}$ be an IFS of contractions on $\mathbb{R}^d$,
- There exists a unique non-empty compact set Λ such that $\Lambda = \bigcup_{i=1}^{m} f_i(\Lambda)$,

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- Let $\Sigma = \{1, \ldots, m\}^{\mathbb{N}}$ be the symbolic space and let $\sigma(i_1, i_2, \ldots) = (i_2, i_3, \ldots)$ the left-shift operator, (denote $\mathbf{i} \wedge \mathbf{j}$ the common part of $\mathbf{i}, \mathbf{j} \in \Sigma$)
- Let $\Pi \colon \Sigma \mapsto \Lambda$ be the natural projection

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- Let $\Pi \colon \Sigma \mapsto \Lambda$ be the natural projection

$$\Pi(\mathbf{i}) = \lim_{n\to\infty} f_{i_1} \circ \cdots \circ f_{i_n}(0),$$

- Let μ be a left-shift invariant, ergodic probability measure on Σ

$$\dim_H \Pi_*\mu =? \text{ and } \Pi_*\mu \overset{?}{\ll} \mathcal{L}_d$$

Conformal systems

- Hutchinson: self-similar systems
- Ruelle: self-conformal systems

Theorem. *If the IFS consists of $C^{1+\alpha}$-conformal mappings (i.e. $D_x f_i \in O(d)$) and strong open set condition holds (i.e. $X \cap U \neq \emptyset$, $f_i(U) \cap f_j(U) = \emptyset$ and $f_i(U) \subseteq U$) then*

$$\dim_H \Lambda = \dim_B \Lambda = s_0, \ \textit{where } s_0 = \inf \left\{ s > 0 : \sum_{n=0}^{\infty} \sum_{\mathbf{i} \in \Sigma_n} \sup_{x \in \Lambda} \|D_x f_{\mathbf{i}}\|^s < \infty \right\},$$

$$\dim_H \Pi_* \mu = \frac{h_\mu}{\chi_\mu}, \ \textit{where } h_\mu = \lim_{n \to \infty} \frac{-1}{n} \sum_{\mathbf{i} \in \Sigma_n} \mu([\mathbf{i}]) \log \mu([\mathbf{i}])$$

$$\textit{and } \chi_\mu = -\int \log \|D_{\Pi(\sigma \mathbf{i})} f_{i_1}\| d\mu(\mathbf{i}).$$

In general, $\overline{\dim}_B \Lambda \leq \min\{d, s_0\}$ and $\overline{\dim}_P \Pi_* \mu \leq \min \left\{ d, \frac{h_\mu}{\chi_\mu} \right\}$.

Conformal systems

Self-similar case ($f_i(x) = \rho_i x + t_i$) dimension

- Pollicott and Simon: $\{\lambda x, \lambda x + 1, \lambda x + 3\}$ of Leb.-a.e. $\lambda \in [1/4, 2/5]$,
- Hochman: under exponential separation and Bernoulli μ,
- Jordan and Rapaport: ergodic measures,
- Varjú: unif. Bernoulli $\{\lambda x, \lambda x + 1\}$ transcendental λ

Self-similar case absolute continuity

- Shmerkin: unif. Bernoulli $\{\lambda x, \lambda x + 1\}$ except 0 dim of $\lambda \in [1/2, 1]$
- Varjú: unif. Bernoulli $\{\lambda x, \lambda x + 1\}$ for some algebraic λ
- Shmerkin and Solomyak: uniform contraction and Bernoulli μ
- Saglietti, Shmerkin and Solomyak: general contracition and Bernoulli μ
- Käenmäki and Orponen: parametrized self-similar IFS with Bernoulli μ

Self-conformal case ($f_i \in C^{1+\alpha}$)

- Simon, Solomyak and Urbański: under transversality condition a.e. parameters
- Peres and Schlag: estimates on the dimension of exceptional parameters
- Hochman and Solomyak: dimension; rational maps (Furstenberg measure)

Non-conformal systems

Let $A \in GL_d(\mathbb{R})$ and let

$$\varphi^s(A) = \begin{cases} \alpha_1(A) \cdots \alpha_{k-1}(A)\alpha_k(A)^{s-k} & \text{if } k-1 \le s < k \text{ for some } k \in \{1, \ldots, d\} \\ |\det(A)|^{s/d} & \text{if } s \ge d, \end{cases}$$

where $\alpha_i(A)$ is the ith singular value of A.

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Let $\{f_1,\ldots,f_m\}$ be an IFS of C^1-contractions on $\mathbb{R}^d$ and let

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Let μ be ergodic σ-invariant on Σ and let

$$\chi_i(\mu) = \lim_{n \to \infty} \frac{-1}{n} \int \log \alpha_i \left(D_{\Pi(\sigma^n \mathbf{i})} f_{\mathbf{i}|_n} \right) d\mu(\mathbf{i}) \text{ and}$$

$$\dim_L \mu = \min\left\{ d, \min_{k=1,\ldots,d} \left\{ k - 1 + \frac{h_\mu - \sum_{i=1}^{k-1} \chi_i(\mu)}{\chi_k(\mu)} \right\} \right\}.$$

Self-affine case

Self-affine case ($\{f_i(x) = A_i x + t_i\}_{i=1}^m$, $A_i \in GL_d(\mathbb{R})$, $t_i \in \mathbb{R}^d$)

- Falconer: $\overline{\dim}_B \Lambda \leq \min\{d, s_0\}$,
- Falconer, Solomyak: if $\|A_i\| < 1/2$ then $\dim_H \Lambda = \dim_B \Lambda = \min\{d, s_0\}$ for Leb.-a.e. $\mathbf{t} = (t_1, \ldots, t_m) \in \mathbb{R}^{dm}$,
- Jordan, Pollicott and Simon: $\dim_H \Pi_* \mu \leq \dim_L \mu$ and if $\|A_i\| < 1/2$ then $\dim_H \Pi_* \mu = \dim_L \mu$ for Leb.-a.e. $\mathbf{t} = (t_1, \ldots, t_m) \in \mathbb{R}^{dm}$.

Self-affine transversality: there exists $C > 0$ s.t. for every $\mathbf{i}, \mathbf{j} \in \Sigma$ with $\mathbf{i} \neq \mathbf{j}$

$$\mathcal{L}_{md}\left(\{\mathbf{t} \in U : \|\Pi_{\mathbf{t}}(\mathbf{i}) - \Pi_{\mathbf{t}}(\mathbf{j})\| < r\}\right) \leq C \min_{k=0,\ldots,d} \left\{ \frac{r^k}{\varphi^k(A_{\mathbf{i}\wedge\mathbf{j}})} \right\}.$$

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- B., Hochman and Rapaport: SOSC and strong irred. on the plane
- Hochman and Rapaport: exponential sep. and strong irred. on the plane
- Rapaport: SOSC and strong irred. for $d = 3$
- Rapaport: diagonal matrices with coordinatewise exponential sep.
- Wu: we will see

General case

- Falconer: $\overline{\dim}_B \Lambda \leq \min\{2, s_0\}$ for planar C^2-contractions with 1-bunched prop.
- Zhang: $\dim_H \Lambda \leq \min\{d, s_0\}$ for C^1-contractions
- Feng and Simon: $\overline{\dim}_B \Lambda \leq \min\{d, s_0\}$ for C^1-contractions
- Jordan and Pollicott: $\overline{\dim}_H \Pi_* \mu \leq \dim_L \mu$ for C^1-contractions
- Feng and Simon: $\overline{\dim}_P \Pi_* \mu \leq \dim_L \mu$ for C^1-contractions

Weierstrass-type functions

Let $m \geq 2$, $\lambda \in (0,1)$ and $\phi\colon \mathbb{R} \mapsto \mathbb{R}$ 1-periodic C^1-map

$$W_{\lambda,m}(x) = \sum_{k=0}^{\infty} \lambda^k \phi(m^k x)$$

The graph$(W_{\lambda,m}) = \{(x, W_{\lambda,m}(x)) : x \in [0,1]\}$ is the attractor of the IFS

$$\left\{ F_i(x,y) = \left(\frac{x+i}{m}, \lambda y + \phi\left(\frac{x+i}{m} \right) \right) \right\}_{i=0,\dots,m-1}$$

In this case, $s_0 = 2 + \dfrac{\log \lambda}{\log m}$.

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- Kaplan, Mallet-Paret and Yorke: either W is C^1 or $\dim_B \mathrm{graph}(W_{\lambda,m}) = s_0$,
- Barański, B. and Romanowska: open and dense set of ϕ in C^3
- Shen: $\phi(x) = \cos(2\pi x)$
- Ren and Shen: for analytic ϕ either W is analytic or $\dim_H \mathrm{graph}(W_{\lambda,m}) = s_0$

The result of Feng and Simon

Let $\mathcal{F}^{\mathbf{t}} = \{F_1^{\mathbf{t}}, \ldots, F_m^{\mathbf{t}}\}$ be a parameterised family of C^1-contractions on $[0,1]^d$ with $\mathbf{t} \in U$ such that $F_i^{\mathbf{t}}([0,1]^d) \subset [0,1]^d$

Definition (Generalised transversality condition (GTC))**.** *There exists* $\psi\colon [0,\infty) \to [0,\infty)$ *with* $\lim_{\delta\to 0} \psi(\delta) = 0$ *such that for every* $\mathbf{t}_0 \in U$ *and* $\delta > 0$ *there exists* $C > 0$ *such that for every* $\mathbf{i}, \mathbf{j} \in \Sigma$ *with* $\mathbf{i} \neq \mathbf{j}$

$$\mathcal{L}(\{\mathbf{t} \in B(\mathbf{t}_0, \delta) : \|\Pi_{\mathbf{t}}(\mathbf{i}) - \Pi_{\mathbf{t}}(\mathbf{j})\| < r\}) \le C e^{|\mathbf{i}\wedge\mathbf{j}|\psi(\delta)} \inf_{\mathbf{k}\in\Sigma} \min_{k=0,\ldots,d} \left\{ \frac{r^k}{\varphi^k(D_{\Pi_{\mathbf{t}_0}(\mathbf{k})} F^{\mathbf{t_0}}_{\mathbf{i}\wedge\mathbf{j}})} \right\}.$$

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Theorem (Feng and Simon)**.** *Let* $\mathcal{F}^{\mathbf{t}} = \{F_1^{\mathbf{t}}, \ldots, F_m^{\mathbf{t}}\}$ *be a parameterised family of* C^1*-contractions on* $[0,1]^d$ *with* $\mathbf{t} \in U$ *satisfying GTC. Then,*

$$\dim_H \Lambda^{\mathbf{t}} = \min\{d, s_0(\mathbf{t})\} \text{ and } \dim_H (\Pi_{\mathbf{t}})_* \mu = \min\{d, \dim_L \mu(\mathbf{t})\}$$

for $\mathcal{L}$*-a.e.* $\mathbf{t} \in U$*. Furthermore,*

$$\mathcal{L}_d(\Lambda^{\mathbf{t}}) > 0 \text{ for a.e. } \mathbf{t} \text{ such that } s_0(\mathbf{t}) > d \text{ and}$$

$$(\Pi_{\mathbf{t}})_* \mu \ll \mathcal{L}_d \text{ for a.e. } \mathbf{t} \text{ such that } \dim_L \mu(\mathbf{t}) > d.$$

The result of Feng and Simon

GTC is verified for the following situations:

- $F_i^{\mathbf{t}} = (f_{i,1}(x_1) + t_{i,1}, \ldots, f_{i,d}(x_d) + t_{i,d})$ and $\sup_{x \in [0,1]} \|f'_{i,k}(x)\| < 1/2$.

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- $F_i^{\mathbf{t}} = (f_{i,1}(x_1) + t_{i,1}, f_{i,2}(x_1, x_2) + t_{i,2}, \ldots, f_{i,d}(x_1, \ldots, x_d) + t_{i,d})$ with

$$\left\|(f_{i,1})'_{x_1}\right\| \geq \left\|(f_{i,2})'_{x_2}\right\| \geq \cdots \geq \left\|(f_{i,d})'_{x_d}\right\|$$

  and $\sup_{x\in[0,1]} \|(f_{i,1})'_{x_1}\| < 1/2$.

Main Setup

Let $\mathcal{F}^{\mathbf{t}} = \{F_1^{\mathbf{t}}, \ldots, F_m^{\mathbf{t}}\}$ be a parameterized family of C^2-contractions on $[0,1]^2$ with $\mathbf{t} = (\mathbf{v}, \mathbf{w}) \in U = V \times W$ such that $F_i^{\mathbf{t}}([0,1]^2) \subset [0,1]^2$

$$F_i^{\mathbf{t}} = (f_i^{\mathbf{v}}(x), g_{i,2}^{\mathbf{v},\mathbf{w}}(x,y)),$$

$$\text{and } 1 > |(g_i^{\mathbf{t}})'_y(x,y)| > |(f_i^{\mathbf{v}})'(x)| > 0 \text{ for every } (x,y) \in [0,1]^2 \text{ and } \mathbf{t} \in \overline{U}.$$

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- For every $\mathbf{i} \in \Sigma$, there exists $u_{\mathbf{i}}^{\mathbf{t}} \colon [0,1]^2 \mapsto \mathbb{R}$ such that for every $j = 1, \ldots, m$

$$D_{(x,y)} F_{i_1}^{\mathbf{t}} \begin{pmatrix} 1 \\ u_{\mathbf{i}}^{\mathbf{t}}(x,y) \end{pmatrix} = (f_{i_1}^{\mathbf{v}})'(x) \begin{pmatrix} 1 \\ u_{\sigma\mathbf{i}}^{\mathbf{t}}\left(F_{i_1}^{\mathbf{t}}(x,y)\right) \end{pmatrix}.$$

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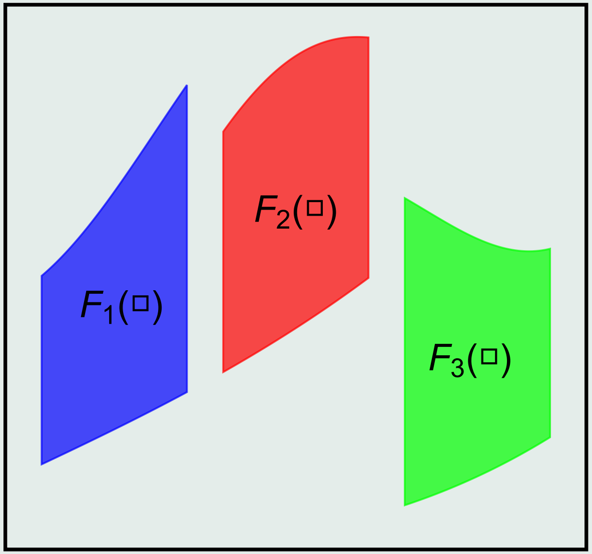
- For every $(x_0, y_0) \in [0,1]^2$ there exists a function $x \mapsto y_{\mathbf{i}}^{\mathbf{t}}((x_0,y_0),x)$ such that

$$\begin{cases} (y_{\mathbf{i}}^{\mathbf{t}})'((x_0,y_0),x) = u_{\mathbf{i}}^{\mathbf{t}}(x, y_{\mathbf{i}}^{\mathbf{t}}((x_0,y_0),x)) \\ y_{\mathbf{i}}^{\mathbf{t}}((x_0,y_0),x_0) = y_0. \end{cases}$$

$$\text{Thus, } g_{i_1}^{\mathbf{t}}(x, y_{\mathbf{i}}^{\mathbf{t}}((x_0,y_0),x)) = (f_{i_1}^{\mathbf{v}}(x), y_{\sigma \mathbf{i}}^{\mathbf{t}}(F_{i_1}^{\mathbf{t}}(x_0,y_0), f_{i_1}^{\mathbf{v}}(x))).$$

Main Setup



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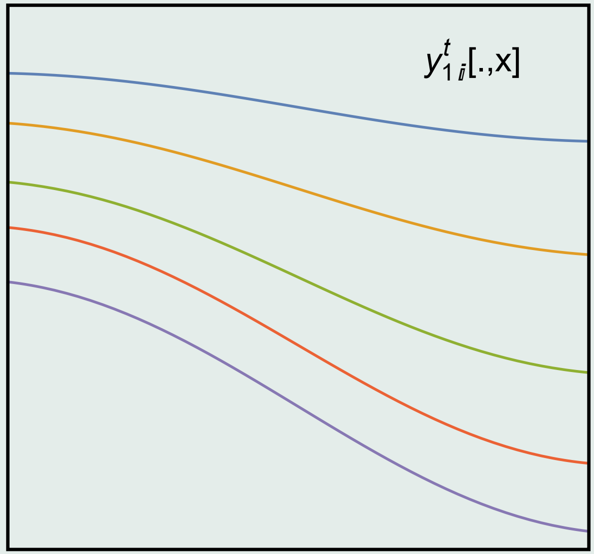

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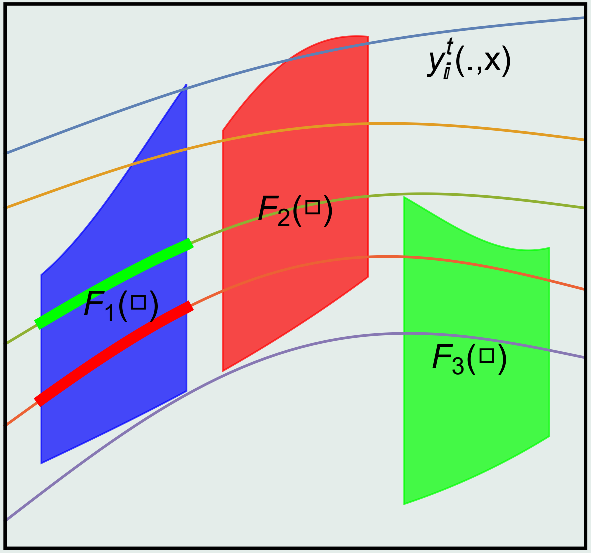

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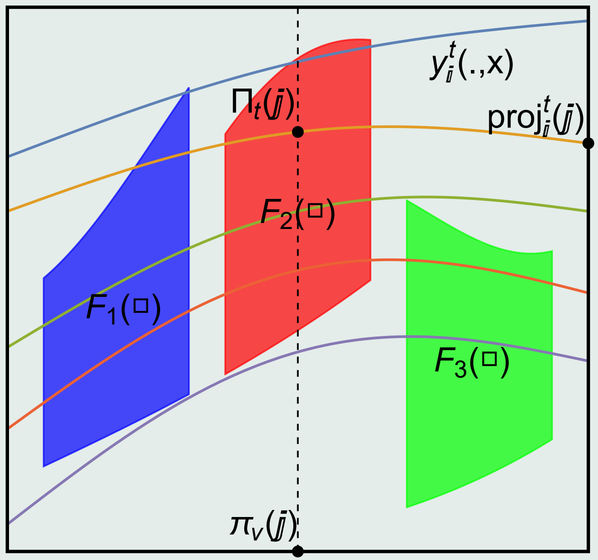

For $(\mathbf{i}, \mathbf{j}) \in \Sigma \times \Sigma$, let $\mathrm{proj}_{\mathbf{i}}^{\mathbf{t}}(\mathbf{j}) = y_{\mathbf{i}}^{\mathbf{t}}(\Pi_{\mathbf{t}}(\mathbf{j}), 0)$.

$\pi_{\mathbf{v}} :=$ the natural projection of the IFS $\{f_i^{\mathbf{v}}\}_{i=1}^{m}$.

Main Result

Let $\mathcal{F}^{\mathbf{t}} = \{F_1^{\mathbf{t}}, \ldots, F_m^{\mathbf{t}}\}$ be a parameterized family of C^2-contractions on $[0,1]^2$ with $\mathbf{t} = (\mathbf{v}, \mathbf{w}) \in U = V \times W$ such that $F_i^{\mathbf{t}}([0,1]^2) \subset [0,1]^2$

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Definition (Triangular Transversality Condition). *There exists $C > 0$ such that for every $\mathbf{i}, \mathbf{j}, \mathbf{k} \in \Sigma$ with $j_1 \neq k_1$*

$$\mathcal{L}\left(\{\mathbf{v} \in V : |\pi_{\mathbf{v}}(\mathbf{j}) - \pi_{\mathbf{v}}(\mathbf{k})| < r\}\right) \leq Cr;$$

$$\mathcal{L}\left(\{\mathbf{w} \in W : |\mathrm{proj}_{\mathbf{i}}^{\mathbf{v}_0,\mathbf{w}}(\mathbf{j}) - \mathrm{proj}_{\mathbf{i}}^{\mathbf{v}_0,\mathbf{w}}(\mathbf{k})| < r\}\right) \leq Cr \textit{ for all } \mathbf{v}_0 \in V.$$

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Theorem (B., Käenmäki)**.** *Let $\mathcal{F}^{\mathbf{t}}$ as above satisfying TTC. Then,*

$$\dim_H \Lambda^{\mathbf{t}} = \min\{d, s_0(\mathbf{t})\} \text{ and } \dim_H (\Pi_{\mathbf{t}})_* \mu = \min\{d, \dim_L \mu(\mathbf{t})\}$$

for $\mathcal{L}$-a.e. $\mathbf{t} \in U$ and quasi-Bernoulli measure μ. Furthermore,

$$\mathcal{L}_d(\Lambda^{\mathbf{t}}) > 0 \text{ for a.e. } \mathbf{t} \text{ such that } s_0(\mathbf{t}) > d \text{ and}$$

$$(\Pi_{\mathbf{t}})_* \mu \ll \mathcal{L}_d \text{ for a.e. } \mathbf{t} \text{ such that } \dim_L \mu(\mathbf{t}) > d.$$

Main Result

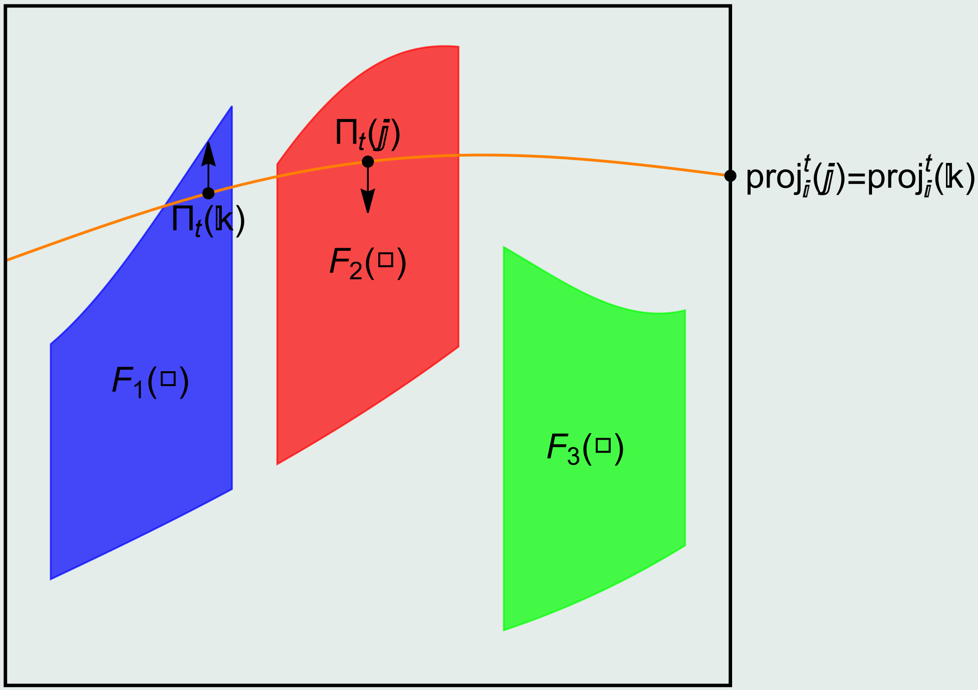
TTC is verified for the following situations:

$$F_i^{\mathbf{t}} = (f_i(x) + t_{i,1}, g_{i,2}(x,y) + t_{i,2}) \text{ such that } F_i^{\mathbf{t}}([0,1]^2) \subset [0,1]^2,$$

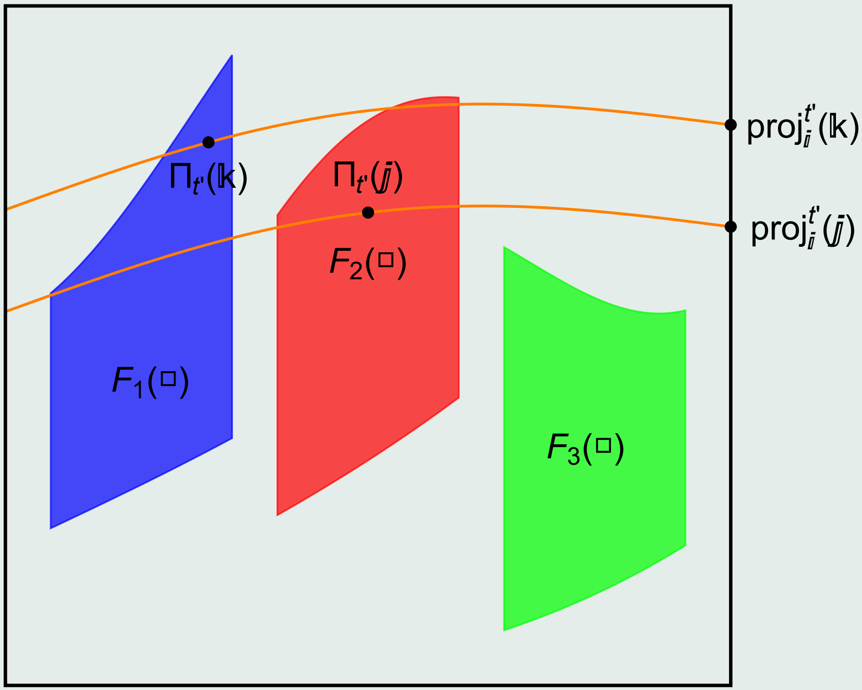
- – $1/2 > \rho > (g_i)'_y(x,y) > f'_i(x) > 0$,
  – $(g_i)''_{xy}(x,y) \leq 0$ and $(g_i)'_x(x,y) \cdot (g_i)''_{yy}(x,y) \geq 0$.

- – $1/4 > |(g_i)'_y(x,y)|$ and $(1/4) \cdot |(g_i)'_y(x,y)| > |f'_i(x)| > 0$,
  – $|(g_i)''_{xy}(x,y)| \leq (1/3) \cdot |(g_i)'_y(x,y)|$,
  – $|(g_i)'_x(x,y) \cdot (g_i)''_{yy}(x,y)| \leq (1/3)|(g_i)'_y(x,y)|^2$.

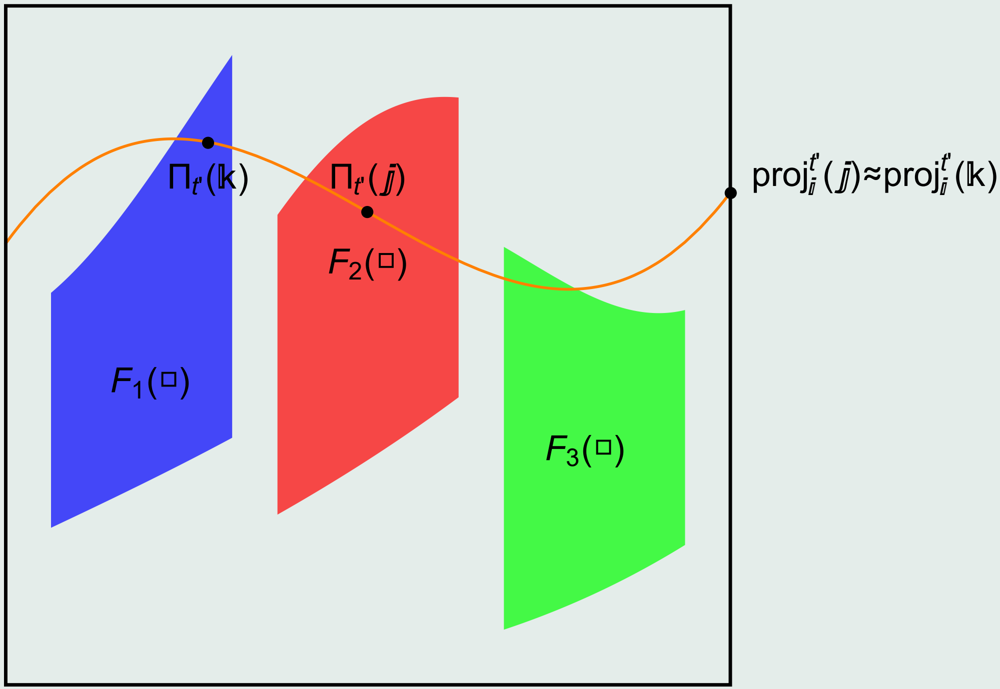
How to show TTC?



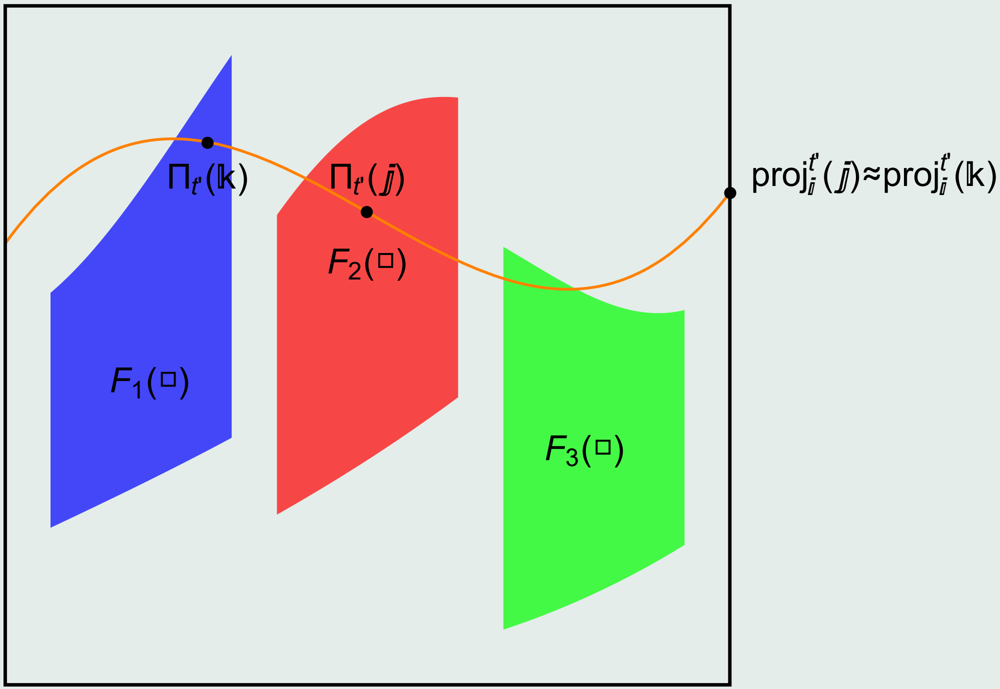
How to show TTC?



How to show TTC?



How to show TTC?



Requires: Strong controll on the dependence of $\mathrm{proj}_{\cdot}^{\mathbf{v},\mathbf{w}}(.)$ on $\mathbf{w}$

TTC implies GTC

$$\mathcal{L}((\mathbf{v},\mathbf{w}) \in B_\delta(\mathbf{v}_0) \times B_\delta(\mathbf{w}_0) : \|\Pi_{\mathbf{v},\mathbf{w}}(\mathbf{j}) - \Pi_{\mathbf{v},\mathbf{w}}(\mathbf{k})\| < r)$$

$$\approx \mathcal{L}((\mathbf{v},\mathbf{w}) \in B_\delta(\mathbf{v}_0) \times B_\delta(\mathbf{w}_0) : |\pi_{\mathbf{v}}(\mathbf{j}) - \pi_{\mathbf{v}}(\mathbf{k})| < r \& |\mathrm{proj}_{\mathbf{i}}^{\mathbf{v},\mathbf{w}}(\mathbf{j}) - \mathrm{proj}_{\mathbf{i}}^{\mathbf{v},\mathbf{w}}(\mathbf{k})| < r)$$

$$= \int \mathbb{1}_{\{\mathbf{v} \in B_\delta(\mathbf{v}_0) : |\pi_{\mathbf{v}}(\mathbf{j}) - \pi_{\mathbf{v}}(\mathbf{k})| < r\}} \mathcal{L}(\mathbf{w} \in B_\delta(\mathbf{w}_0) : |\mathrm{proj}_{\mathbf{i}}^{\mathbf{v},\mathbf{w}}(\mathbf{j}) - \mathrm{proj}_{\mathbf{i}}^{\mathbf{v},\mathbf{w}}(\mathbf{k})| < r) d\mathbf{v}$$

TTC implies GTC

$$\mathcal{L}((\mathbf{v},\mathbf{w}) \in B_\delta(\mathbf{v}_0)\times B_\delta(\mathbf{w}_0) : \|\Pi_{\mathbf{v},\mathbf{w}}(\mathbf{j}) - \Pi_{\mathbf{v},\mathbf{w}}(\mathbf{k})\| < r)$$
$$\approx \mathcal{L}((\mathbf{v},\mathbf{w}) \in B_\delta(\mathbf{v}_0)\times B_\delta(\mathbf{w}_0) : |\pi_{\mathbf{v}}(\mathbf{j}) - \pi_{\mathbf{v}}(\mathbf{k})| < r \& |\mathrm{proj}_{\mathbf{i}}^{\mathbf{v},\mathbf{w}}(\mathbf{j}) - \mathrm{proj}_{\mathbf{i}}^{\mathbf{v},\mathbf{w}}(\mathbf{k})| < r)$$
$$= \int \mathbb{1}_{\{\mathbf{v}\in B_\delta(\mathbf{v}_0) : |\pi_{\mathbf{v}}(\mathbf{j})-\pi_{\mathbf{v}}(\mathbf{k})|<r\}} \mathcal{L}(\mathbf{w} \in B_\delta(\mathbf{w}_0) : |\mathrm{proj}_{\mathbf{i}}^{\mathbf{v},\mathbf{w}}(\mathbf{j}) - \mathrm{proj}_{\mathbf{i}}^{\mathbf{v},\mathbf{w}}(\mathbf{k})| < r) d\mathbf{v}$$

$$\begin{aligned}
|\mathrm{proj}_{\mathbf{i}}^{\mathbf{t}}(\mathbf{j}) - \mathrm{proj}_{\mathbf{i}}^{\mathbf{t}}(\mathbf{k})| &\approx |y_{\mathbf{i}}(\Pi_{\mathbf{t}}(\mathbf{j}), f_{\mathbf{j}\wedge\mathbf{k}}(0)) - y_{\mathbf{i}}(\Pi_{\mathbf{t}}(\mathbf{k}), f_{\mathbf{j}\wedge\mathbf{k}}(0))| \\
&= \left| g_{\mathbf{j}\wedge\mathbf{k}}^{\mathbf{v},\mathbf{w}}(0, \mathrm{proj}_{\overleftarrow{\mathbf{j}\wedge\mathbf{k}\mathbf{i}}}^{\mathbf{t}}(\sigma^{|\mathbf{j}\wedge\mathbf{k}|}\mathbf{j})) - g_{\mathbf{j}\wedge\mathbf{k}}^{\mathbf{v},\mathbf{w}}(0, \mathrm{proj}_{\overleftarrow{\mathbf{j}\wedge\mathbf{k}\mathbf{i}}}^{\mathbf{t}}(\sigma^{|\mathbf{j}\wedge\mathbf{k}|}\mathbf{k})) \right| \\
&\geq e^{-\psi(\delta)|\mathbf{j}\wedge\mathbf{k}|} |(g_{\mathbf{j}\wedge\mathbf{k}}^{\mathbf{v_0},\mathbf{w_0}})'_y(0,0)| \cdot |\mathrm{proj}_{\overleftarrow{\mathbf{j}\wedge\mathbf{k}\mathbf{i}}}^{\mathbf{t}}(\sigma^{|\mathbf{j}\wedge\mathbf{k}|}\mathbf{j}) - \mathrm{proj}_{\overleftarrow{\mathbf{j}\wedge\mathbf{k}\mathbf{i}}}^{\mathbf{t}}(\sigma^{|\mathbf{j}\wedge\mathbf{k}|}\mathbf{k})|
\end{aligned}$$

TTC implies GTC

$$
\begin{aligned}
&\mathcal{L}((\mathbf{v},\mathbf{w}) \in B_\delta(\mathbf{v}_0) \times B_\delta(\mathbf{w}_0) : \|\Pi_{\mathbf{v},\mathbf{w}}(\mathbf{j}) - \Pi_{\mathbf{v},\mathbf{w}}(\mathbf{k})\| < r) \\
&\approx \mathcal{L}((\mathbf{v},\mathbf{w}) \in B_\delta(\mathbf{v}_0) \times B_\delta(\mathbf{w}_0) : |\pi_{\mathbf{v}}(\mathbf{j}) - \pi_{\mathbf{v}}(\mathbf{k})| < r \& |\mathrm{proj}_{\mathbf{i}}^{\mathbf{v},\mathbf{w}}(\mathbf{j}) - \mathrm{proj}_{\mathbf{i}}^{\mathbf{v},\mathbf{w}}(\mathbf{k})| < r) \\
&= \int \mathbb{1}_{\{\mathbf{v}\in B_\delta(\mathbf{v}_0) : |\pi_{\mathbf{v}}(\mathbf{j}) - \pi_{\mathbf{v}}(\mathbf{k})| < r\}} \mathcal{L}(\mathbf{w} \in B_\delta(\mathbf{w}_0) : |\mathrm{proj}_{\mathbf{i}}^{\mathbf{v},\mathbf{w}}(\mathbf{j}) - \mathrm{proj}_{\mathbf{i}}^{\mathbf{v},\mathbf{w}}(\mathbf{k})| < r) d\mathbf{v} \\
&\le C e^{\psi(\delta)|\mathbf{j}\wedge\mathbf{k}|} \int \mathbb{1}_{\{\mathbf{v}\in B_\delta(\mathbf{v}_0) : |\pi_{\mathbf{v}}(\mathbf{j}) - \pi_{\mathbf{v}}(\mathbf{k})| < r\}} \min\left\{1, \frac{r}{|(g_{\mathbf{j}\wedge\mathbf{k}}^{\mathbf{v_0},\mathbf{w_0}})'_y(0,0)|}\right\} d\mathbf{v} \\
&\le C e^{\psi(\delta)|\mathbf{j}\wedge\mathbf{k}|} \min\left\{1, \frac{r}{|(g_{\mathbf{j}\wedge\mathbf{k}}^{\mathbf{v_0},\mathbf{w_0}})'_y(0,0)|}\right\} \int \mathbb{1}_{\left\{\mathbf{v}\in B_\delta(\mathbf{v}_0) : |\pi_{\mathbf{v}}(\sigma^{|\mathbf{j}\wedge\mathbf{k}|}\mathbf{j}) - \pi_{\mathbf{v}}(\sigma^{|\mathbf{j}\wedge\mathbf{k}|}\mathbf{k})| < \frac{e^{\psi(\delta)} r}{|(f_{\mathbf{j}\wedge\mathbf{k}}^{\mathbf{v}_0})'(0)|}\right\}} d\mathbf{v} \\
&\le C^2 e^{2\psi(\delta)|\mathbf{j}\wedge\mathbf{k}|} \min\left\{1, \frac{r}{|(g_{\mathbf{j}\wedge\mathbf{k}}^{\mathbf{v_0},\mathbf{w_0}})'_y(0,0)|}\right\} \min\left\{1, \frac{r}{|(f_{\mathbf{j}\wedge\mathbf{k}}^{\mathbf{v}_0})'(0)|}\right\}.
\end{aligned}
$$

weak Ledrappier-Young formula

Let $\mathcal{F} = \{F_1, \ldots, F_m\}$ be a parameterized family of C^2-contractions on $[0,1]^2$ such that $F_i([0,1]^2) \subset [0,1]^2$,

$$F_i = (f_i(x), g_{i,2}(x,y)),$$

$$\text{and } 1 > |(g_i)'_y(x,y)| > |(f_i)'(x)| > 0 \text{ for every } (x,y) \in [0,1]^2.$$

Theorem (weak Ledrappier-Young formula)**.** *Let μ be a quasi-Bernoulli ergodic σ-invariant measure, and $\overleftarrow{\mu}$ its reversed measure, that is $\overleftarrow{\mu}([\mathbf{i}]) = \mu([\overleftarrow{\mathbf{i}}])$.*
For $\overleftarrow{\mu}$-a.e. $\mathbf{i}$

$$\dim_H(\Pi)_*\mu \geq \frac{h_\mu - h(\mu|\Pi)}{\chi_2(\mu)} + \left(1 - \frac{\chi_1(\mu)}{\chi_2(\mu)}\right) \dim_H(\mathrm{proj}_{\mathbf{i}})_*\mu,$$

where

$$\chi_1^{\mathbf{t}}(\mu) = -\int \log|(g_{i_0})'_y(\Pi(\sigma\mathbf{i}))| d\mu(\mathbf{i}),\ \ \chi_2^{\mathbf{t}}(\mu) = -\int \log|f'_{i_0}(\Pi(\sigma\mathbf{i}))| d\mu(\mathbf{i}),$$

$$h(\mu|\Pi_{\mathbf{t}}) = -\int \log \mu_{\mathbf{i}}^{\Pi_{\mathbf{t}}^{-1}}([i_0])\mu(\mathbf{i}).$$

Transversality

Let $\mathcal{F}^{\mathbf{t}} = \{F_1^{\mathbf{t}}, \dots, F_m^{\mathbf{t}}\}$ be a parameterized family of C^2-contractions on $[0,1]^2$ with $\mathbf{t} = (\mathbf{v}, \mathbf{w}) \in U = V \times W$ such that $F_i^{\mathbf{t}}([0,1]^2) \subset [0,1]^2$

$$F_i^{\mathbf{t}} = (f_i^{\mathbf{v}}(x), g_{i,2}^{\mathbf{v},\mathbf{w}}(x,y)),$$

Recall that for $\mathbf{i}, \mathbf{j}, \mathbf{k} \in \Sigma$ with $j_1 \neq k_1$

$$\mathcal{L}\left(\{\mathbf{v} \in V : |\pi_{\mathbf{v}}(\mathbf{j}) - \pi_{\mathbf{v}}(\mathbf{k})| < r\}\right) \leq Cr;$$

$$\mathcal{L}\left(\{\mathbf{w} \in W : |\mathrm{proj}_{\mathbf{i}}^{\mathbf{v}_0,\mathbf{w}}(\mathbf{j}) - \mathrm{proj}_{\mathbf{i}}^{\mathbf{v}_0,\mathbf{w}}(\mathbf{k})| < r\}\right) \leq Cr \text{ for all } \mathbf{v}_0 \in V.$$

Using Solomyak's Generalised Projection Scheme Theorem

$$\dim_H(\mathrm{proj}_{\mathbf{i}}^{\mathbf{t}})_*\mu = \min\left\{1, \frac{h_\mu}{\chi_1^{\mathbf{t}}(\mu)}\right\} \text{ for } \mathcal{L} \times \overleftarrow{\mu}\text{-a.e. } (\mathbf{t}, \mathbf{i}).$$

Transversality

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If $h_\mu < \chi_1^{\mathbf{t}}(\mu) + \chi_2^{\mathbf{t}}(\mu)$ then using the method of Barański, Gutman and Śpiewak

$$\#\Pi_{\mathbf{t}}^{-1}(\Pi_{\mathbf{t}}(\mathbf{i})) = 1 \text{ for } \mathcal{L}_m \times \overleftarrow{\mu}\text{-a.e. } (\mathbf{t}, \mathbf{i}) \Rightarrow h(\mu|\Pi_{\mathbf{t}}) = 0.$$

The absolutely continuous regime

Let $\mathcal{F}^{\mathbf{t}} = \{F_1^{\mathbf{t}}, \ldots, F_m^{\mathbf{t}}\}$ be a parameterized family of C^2-contractions on $[0,1]^2$ with $\mathbf{t} = (\mathbf{v}, \mathbf{w}) \in U = V \times W$ such that $F_i^{\mathbf{t}}([0,1]^2) \subset [0,1]^2$

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By Simon, Solomyak and Urbański,

$$(\pi_{\mathbf{v}})_*\mu \ll \mathcal{L} \text{ for } \mathcal{L}\text{-a.e. } \mathbf{v} \in V.$$

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Let $\mu_x^{(\pi_{\mathbf{v}}^1)^{-1}}$:=the conditional measure of μ on $(\pi_{\mathbf{v}})^{-1}(x)$, and by Feng-Hu's Theorem $h(\mu_x^{(\pi_{\mathbf{v}})^{-1}}) = h_\mu - \chi_2^{\mathbf{v}}(\mu)$ for $(\pi_{\mathbf{v}})_*\mu$-a.e. x.

The absolutely continuous regime

Let $\mathcal{F}^{\mathbf{t}} = \{F_1^{\mathbf{t}}, \dots, F_m^{\mathbf{t}}\}$ be a parameterized family of C^2-contractions on $[0,1]^2$ with $\mathbf{t} = (\mathbf{v}, \mathbf{w}) \in U = V \times W$ such that $F_i^{\mathbf{t}}([0,1]^2) \subset [0,1]^2$

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Using the generalised projection scheme again, since $\frac{h_\mu - \chi_2^{\mathbf{v}}(\mu)}{\chi_1^{\mathbf{t}}(\mu)} > 1$

$$(\Pi_{\mathbf{t}})_*\mu_x^{(\pi_{\mathbf{v}})^{-1}} \ll \mathcal{L} \text{ for } \mathcal{L}\text{-a.e. } \mathbf{t}.$$

Thank you for your attention!