Multiplicative Markoff-Lagrange spectrum and symbolic dynamics

Shigeki Akiyama (Univ. Tsukuba)

HongKong 11-15 Dec 2023

ArXiv:2204.10510

Joint work with Teturo Kamae and Hajime Kaneko

Lagrange Spectrum

Let $\alpha \in \mathbb{R} \setminus \mathbb{Z}$. Hurwitz showed that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}$$

has infinitely many rational solutions p/q. The equality is attained if and only if α is tail equivalent to $(1+\sqrt{5})/2$ in their continued fractions. Excluding such numbers,

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{8}q^2}$$

has infinitely many rational solutions. The equality is attained if and only if α is tail equivalent to $1+\sqrt{2}$. Next we have $\sqrt{221}/5$ and so on. Therefore, we are interested in the Lagrange spectrum:

$$\mathcal{L} = \left\{ \limsup_n \frac{1}{n\|n\alpha\|} \;\middle|\; \alpha \in \mathbb{R} \setminus \mathbb{Q} \right\}.$$

1. $\mathcal{L}$ is closed (Cusick).

2. $\mathcal{L} \supset [6, \infty)$ (Hall). 6 is replaced by Freiman constant:

$$\frac{2221564096 + 283748\sqrt{462}}{491993569} = 4.5278295661608 7914 \ldots.$$

3. $\mathcal{L} \cap [0,3) = \left\{ \frac{\sqrt{9m^2-4}}{m} \;\middle|\; m = 1, 2, 5, \ldots \right\}$ where m are Markoff numbers appear as integer solutions of $x^2 + y^2 + z^2 = 3xyz$. 3 is the minimum accumulation point (Markoff).

The key is known as Perron formula

$$\mathcal{L} = \left\{ \limsup_{n} [a_n; a_{n+1}, a_{n+2}, \ldots] + [0; a_{n-1}, a_{n-2}, \ldots] \;\middle|\; (a_n) \in \mathbb{N}^{\mathbb{Z}} \right\}$$

which transfer the problem to the optimization problem on the shift space.

Multiplicative version in Pisot base

In the previous paper [1] with H. Kaneko, for a Pisot number $\alpha > 1$ we showed that

$$\mathcal{L}(\alpha) = \left\{ \limsup_{n\to\infty} \|\xi\alpha^n\| \,\middle|\, \xi \in \mathbb{R} \right\}$$

is closed. Further we derived a complete description of the minimum accumulation point and the discrete points below when α is an integer > 1 or quadratic Pisot unit > 3. When α is an integer, Lebesgue measure of $\mathcal{L}(\alpha)$ is zero. If α is a quadratic Pisot unit > 3, there exists $v < 1/2$ that $\mathcal{L}(\alpha) \supset [v, 1/2]$.

The key is the formula that lift the problem to a symbolic dynamical setting. Here is a schematic view:

$$\begin{array}{ccc} \Omega & \xrightarrow{\sigma} & \Omega \\ h\Big\downarrow & & h\Big\downarrow \\ \xi\alpha^n \pmod{\mathbb{Z}} & \xrightarrow{\times\alpha} & \xi\alpha^{n+1} \pmod{\mathbb{Z}} \end{array}$$

where Ω is a shift space and $\sigma((t_m)) = (t_{m+1})$ is the shift map acting on bi-infinite sequences $(t_m)_{m\in\mathbb{Z}}$ over a finite number of symbols.

Note that there is no appropriate dynamical system for $\times\alpha$. Beta expansion works only when $\alpha \in \mathbb{Z}$.

Example: Quadratic Pisot unit case

Theorem 1. *Let* $\alpha = (b + \sqrt{b^2 - 4})/2 > 3$ *and* $\alpha_2 = (b - \sqrt{b^2 - 4})/2$. *Consider the* **negative continued fraction** *expansion*

$$\frac{1}{1+\alpha} = \cfrac{1}{1+b-\cfrac{1}{b-\cfrac{1}{b-\cfrac{1}{\ddots}}}} =: [0; 1+b, b, b, b, \dots]_{neg}.$$

Using its n*-th convergent* $P_n/Q_n = [0, \underbrace{1+b, b, b, \dots, b}_{n}]_{neg}$, *we define*

$$\frac{p_{2n}}{q_{2n}} = \frac{P_n}{Q_n} \quad \textit{and} \quad \frac{p_{2n-1}}{q_{2n-1}} = \frac{P_{n-1}+P_n}{Q_{n-1}+Q_n}.$$

Then $(p_n/q_n)_{n=0,1,\dots}$ *is a strictly increasing sequence converging to* $1/(1+\alpha)$ *and we have*

$$\left\{ \limsup_{n\to\infty} \|\xi\alpha^n\| \;\middle|\; \xi \in \mathbb{R} \right\} \cap \left[0, \frac{1}{1+\alpha}\right] = \left\{ \frac{p_n}{q_n} \;\middle|\; n = 0, 1, 2, \dots \right\} \bigcup \left\{ \frac{1}{1+\alpha} \right\}.$$

Moreover, $X_n := \{\xi \in \mathbb{R} \mid \limsup_{n\to\infty} \|\xi\alpha^n\| = p_n/q_n\}$ *is a subset of* $\mathbb{Q}(\alpha)$*, explicitly described in terms of Christoffel words, and*

$$X_\infty := \left\{ \xi \in \mathbb{R} \;\middle|\; \limsup_{n\to\infty} \|\xi\alpha^n\| = \frac{1}{1+\alpha} \right\}$$

is an uncountable set, described by eventually balanced words, a generalization of Sturmian words.

Multiplicative version in Linear recurrence

Let $f(X) \in \mathbb{Z}[X]$ be a hyperbolic polynomial. We are interested in the set $\Xi(f)$ consisting of all complex sequences (x_n) which satisfy a fixed linear recurrence whose characteristic polynomial is $f(X) = \sum_{i=0}^{d} A_i X^i$, i.e,,

$$\Xi(f) = \left\{ (x_n)_{n \in \mathbb{Z}} \;\middle|\; \sum_{i=0}^{d} A_i x_{n+i} = 0 \right\}.$$

and set

$$\mathcal{L}(f) := \left\{ \limsup_{n \to \infty} \|\mathrm{Re}(x_n)\| \;\middle|\; x_n \in \Xi(f) \right\}.$$

Without loss of generality, one assume that $x_{-n} \to 0$ for $n \to \infty$.

Symbolic Dynamics related to this problem

Let $x = u(x) + \varepsilon(x)$ where $u(x) \in \mathbb{Z}$ and $\varepsilon(x) \in [-1/2, 1/2)$. Given monic hyperbolic polynomial $f(X) = \sum_{i=0}^{d} A_i X^i$ and $(y_n) \in \Xi(f)$. Define $x_n = \mathrm{Re}(y_n)$ and

$$s(x_n) = \sum_{i=0}^{d} A_i u(x_{n+i}) = -\sum_{i=0}^{d} A_i \varepsilon(x_{n+i}). \qquad (1)$$

Note that $s(x_n) \in [-m, m] \cap \mathbb{Z}$ with $m = (\sum_{i=0}^{d} |A_i|)/2$.

We wish to prove that there exists a bi-infinite real sequence $(\rho_n^f)_{n\in\mathbb{Z}}$ that decreases exponentially as $n \to \pm\infty$ that

$$\varepsilon(x_n) = \sum_{n\in\mathbb{Z}} s(x_n)\rho_{-n}^f.$$

Conversely for ANY bounded sequence s_n that $s_{-m} = 0$ for sufficiently large m, then there exists $(x_n) \in \mathrm{Re}(\Xi(f))$ that

$$\varepsilon(x_n) \equiv \sum_{n\in\mathbb{Z}} s_n\rho_{-n}^f \pmod{\mathbb{Z}}.$$

We consider square matrices $\mathcal{A} = (a_{m,n})_{m,n\in\mathbb{Z}}, \mathcal{B} = (b_{m,n})_{m,n\in\mathbb{Z}}$ with the index set $\mathbb{Z} \times \mathbb{Z}$. When $\sum_{j\in\mathbb{Z}} a_{m,j} b_{j,n}$ converge for any $m, n \in \mathbb{Z}$, we define the product of $\mathcal{A}, \mathcal{B}$ by

$$(a_{m,j})_{m,j\in\mathbb{Z}}(b_{j,n})_{m,n\in\mathbb{Z}} = \left(\sum_{j\in\mathbb{Z}} a_{m,j} b_{j,n}\right)_{m,n\in\mathbb{Z}}. \qquad (2)$$

We call that $\mathcal{B}$ an inverse matrix of $\mathcal{A}$ if both of $\mathcal{AB}$ and $\mathcal{BA}$ are defined and if

$$\mathcal{AB} = \mathcal{BA} = (\mathbf{1}_{m=n})_{m,n\in\mathbb{Z}}.$$

We also consider the column vector $\boldsymbol{y} = (y_m)_{m\in\mathbb{Z}}$ with the index set $\mathbb{Z}$. When $\sum_{j\in\mathbb{Z}} a_{m,j} y_j$ converges for any $m \in \mathbb{Z}$, then we define the product

$\mathcal{A}\boldsymbol{y}$ by

$$(a_{m,j})_{m,j\in\mathbb{Z}}(y_j)_{j\in\mathbb{Z}} = \left(\sum_{j\in\mathbb{Z}} a_{m,j}y_j\right)_{m\in\mathbb{Z}}. \tag{3}$$

Note that even if both of $(\mathcal{A}\mathcal{B})\boldsymbol{y}$ and $\mathcal{A}(\mathcal{B}\boldsymbol{y})$ are defined, these two vectors do not necessarily coincide with each other.

We consider the column vector $\boldsymbol{\epsilon} = (\epsilon_n)_{n\in\mathbb{Z}} \in \boldsymbol{E}_f$. We define a column vector $\boldsymbol{s} = (s_m)_{n\in\mathbb{Z}}$ by $\boldsymbol{s} = \mathcal{A}_f\boldsymbol{\epsilon}$, where the square matrix $\mathcal{A}_f$ with the index set $\mathbb{Z}\times\mathbb{Z}$ is defined as follows:

$$\mathcal{A}_f := (-A_{n-m}\mathbf{1}_{0\leq n-m\leq D})_{m,n\in\mathbb{Z}}.$$

This is a matrix representation of the relation (1):

$$s_m = -\sum_{n=m}^{m+D} A_{n-m}\epsilon_n = \sum_{n\in\mathbb{Z}} (-A_{n-m}\mathbf{1}_{0\le n-m\le D})\epsilon_n.$$

We shall represent $\boldsymbol{\epsilon}$, using the inverse matrix of $\mathcal{A}_f$. For any integer n, put

$$\rho_n^f = \frac{-1}{2\pi i}\int_{|z|=1+0} \frac{z^{n-1}}{f(z)} dz,$$

where the integral is around the unit circle to the positive direction, avoiding poles, if exist, from outside. It is easily seen that ρ_n^f is a real number for any $n \in \mathbb{Z}$. We now define the square matrix $\mathcal{B}_f$ with the index set $\mathbb{Z}\times\mathbb{Z}$ by

$$\mathcal{B}_f = (\rho_{m-n}^f)_{m,n\in\mathbb{Z}}. \tag{4}$$

Theorem 2. *(1)* $\mathcal{B}_f$ *is the inverse matrix of* $\mathcal{A}_f$ *if the domain is restricted to* $\mathcal{A}_f \boldsymbol{E}_f$*.*
(2) If (s_m) *is defined in (1) with respect to* (ϵ_m)*, then we have*

$$(\epsilon_m) = \mathcal{B}_f(s_m).$$

(3) For any $(t_m) \in \Omega_0$*, there exists* $(\epsilon_m) \in \boldsymbol{E}_f$ *such that*

$$\mathcal{B}_f(t_m) \equiv (\epsilon_m) \pmod{\mathbb{Z}}.$$

Lemma 3. *For any integer* n*, we have*

$$\sum_{j=0}^{D} -A_j \rho_{j+n}^f = \mathbf{1}_{n=0}.$$

Proof.

$$\sum_{j=0}^{D} -A_j \rho^f_{j+n} = \sum_{j=0}^{D} A_j \frac{1}{2\pi i} \int_{|z|=1+0} \frac{z^{j+n-1}}{f(z)} dz$$

$$= \frac{1}{2\pi i} \int_{|z|=1+0} \frac{\sum_{j=0}^{D} A_j z^j}{f(z)} z^{n-1} dz$$

$$= \frac{1}{2\pi i} \int_{|z|=1+0} z^{n-1} dz = \mathbf{1}_{n=0}.$$

□

Lemma 4. *If $f(X)$ is monic, then*

$$\operatorname{Res}\left(\frac{z^{n-1}}{f(z)}, \infty\right) \in \mathbb{Z}.$$

Results

Theorem 5. *Suppose that* $f \in \mathbb{Z}[X]$ *is monic and hyperbolic. Then the set* $\mathcal{L}(f)$ *is closed in* $[0, 1/2]$.

Theorem 6. *Suppose that* f *is monic and hyperbolic. Then 0 is an isolated point of the set* $\mathcal{L}(f)$. *If* f *is monic and not hyperbolic, then* $\mathcal{L}(f)$ *is dense in* $[0, 1/2]$.

Theorem 7. *Suppose that* $f \in \mathbb{Z}[X]$ *is monic and hyperbolic. Let*

$$\beta_1 := \min\{|\alpha_1|, |\alpha_2|, \ldots, |\alpha_p|\}$$

$$\beta_2 := \min\{|\alpha_{p+1}|^{-1}, \ldots, |\alpha_d|^{-1}\}$$

$$\beta := \min\{\beta_1, \beta_2\}$$

(1) $1/2$ *is an accumulation point of* $\mathcal{L}(f)$.

(2) Let f *be separable. Assume that either* $\beta_1 = \beta$ *is attained by* $|\alpha_i|$ *at*

a single index $i = 1, \ldots, p$, *or* $\beta_2 = \beta$ *is attained by* $|\alpha_i|^{-1}$ *at a single index* $i = p+1, \ldots, d$, *and*

$$\left\lceil \frac{\beta - 1}{2} \right\rceil \sum_{n \in \mathbb{Z}} |\rho_n^f| < \frac{1}{2} \tag{5}$$

holds, then there exists a sequence of proper intervals $[c_i, d_i]$ *contained in* $\mathcal{L}(f)$ *and* $\lim_{i \to \infty} c_i = 1/2$.

(3) Let f *be separable. Assume that* β_1 *is attained by* $|\alpha_i|$ *at a single index* $i = 1, \ldots, p$ *and that* β_2 *is attained by* $|\alpha_i|^{-1}$ *at a single index* $i = p+1, \ldots, d$. *Moreover, suppose that*

$$\left\lceil \frac{\widetilde{\beta} - 1}{2} \right\rceil \sum_{n \in \mathbb{Z}} |\rho_n^f| < \frac{1}{2}, \tag{6}$$

where $\widetilde{\beta} := \max\{\beta_1, \beta_2\}$. *Then there exists a* $v < 1/2$ *such that* $[v, 1/2] \subset$

$\mathcal{L}(f)$.

Let

$$E^{(k)}(X) := \frac{1 + X^{2^k} - (1-X)\prod_{m=0}^{k-1}(1 - X^{2^m})}{2X(1 + X^{2^k})}$$

for $k \geq 0$. Moreover, for any $r \geq 1$, put

$$E_r^{(k)}(X_1, \ldots, X_r) := \sum_{i=1}^{r} \left(\prod_{\substack{1 \leq j \leq r \\ j \neq i}} \frac{X_i}{X_i - X_j} \right) E^{(k)}(X_i).$$

Theorem 8. *Assume that* $f(X)$ *is monic and that* $\alpha_j > 1$ *for any* $1 \leq$

$j \le d$. Moreover, suppose that

$$\alpha_1^{-1} + \alpha_2^{-1} + \cdots + \alpha_d^{-1} \le \frac{1}{2}. \tag{7}$$

Then $|a_0|^{-1} E_d^{(\infty)}(\alpha_1^{-1}, \ldots, \alpha_d^{-1})$ is the minimal limit point of $\mathcal{L}(f(X))$. Moreover, we have

$$\mathcal{L}(f(X)) \cap \left(0, \frac{1}{|a_0|} E_d^{(\infty)}(\alpha_1^{-1}, \ldots, \alpha_d^{-1})\right)$$
$$= \left\{ \frac{1}{|a_0|} E_d^{(k)}(\alpha_1^{-1}, \ldots, \alpha_d^{-1}) \;\middle|\; k = 0, 1, \ldots \right\}.$$

Examples

Let

$$f(X) = X^3 + 2X^2 + 6X - 2$$

and k be a non negative integer. Then $f(X)$ has two complex roots $\alpha, \overline{\alpha}$ and a real root β with $\alpha \approx -1.1495 + 2.3165\sqrt{-1}$ and $\beta \approx 0.2991$. The set

$$\mathcal{L}_1 = \left\{ \limsup_{n\to\infty} \|\mathrm{Re}(p(n)\alpha^n)\| \;\middle|\; p(x) \in \mathbb{C}[x],\ \deg(p) \le k \right\}$$

is closed by Theorem 5. By Theorem 6, the point 0 is an isolated point. One can confirm (5), and thus there exists proper intervals in $\mathcal{L}_1$ whose end points converge to $1/2$ by Theorem 7. Moreover, by using fractal geometry discussion, we can find an interval $[1/2 - \varepsilon, 1/2] \subset \mathcal{L}_1$ with $\varepsilon > 0$.

For $f(X) = X^2 - 20X + 82$, the set

$$\mathcal{L}_2 = \left\{ \limsup_{n\to\infty} \left\| \xi_1 (10+3\sqrt{2})^n + \xi_2 (10-3\sqrt{2})^n \right\| \;\middle|\; \xi_1, \xi_2 \in \mathbb{R} \right\}$$

is closed and 0 is an isolated point. Moreover, Theorem 8 gives a very precise description of the discrete part of the spectrum below the smallest accumulation point. Also one can confirm (5) and there are proper intervals in $\mathcal{L}_2$ whose endpoints converge to $1/2$.

References

[1] S. Akiyama and H. Kaneko, *Multiplicative analogue of Markoff-Lagrange spectrum and Pisot numbers*, Adv. Math. **380** (2021), Paper No. 107547, 39, Corrigendum : Paper No. 107996, 2.