Subwavelength localized modes for acoustic waves

Sanghyeon Yu

Korea University

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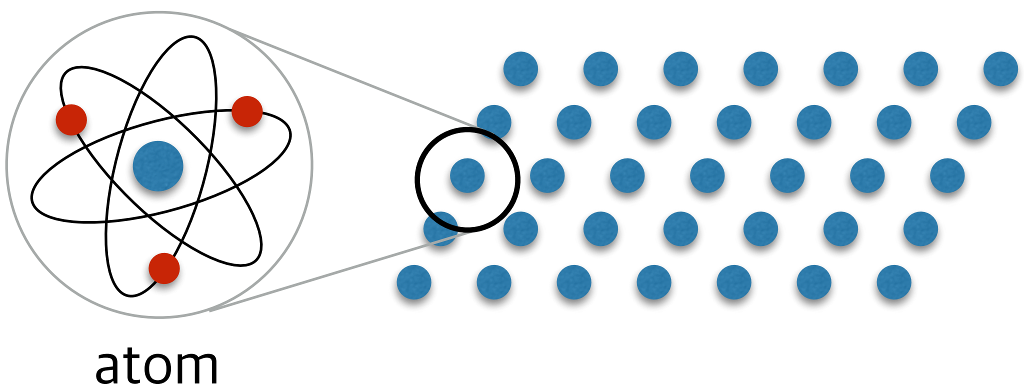
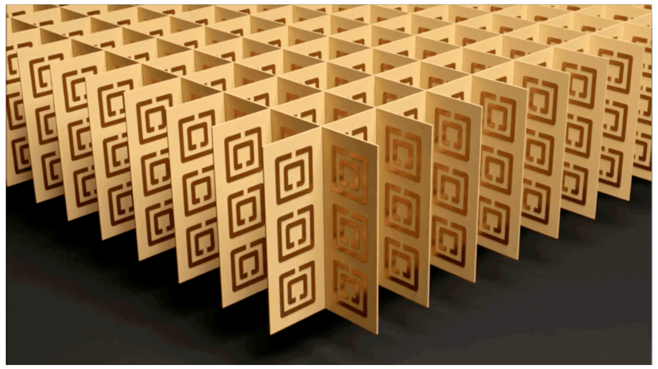
Meta-material

- A Material made of artificially designed atoms
- exhibit new material properties that cannot be found in nature.

  (the Greek word 'meta' means beyond.)
- A new paradigm of materials sciences.



Ordinary Materials
(made of atoms)



Meta-materials
(made of small artificial structures)

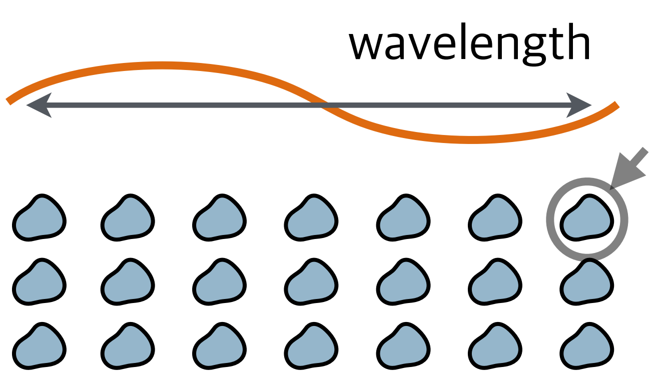
figures from wikipedia.org

Designing Meta-materials

- Controlling physical waves in subwavelength scales overcoming the diffraction limit

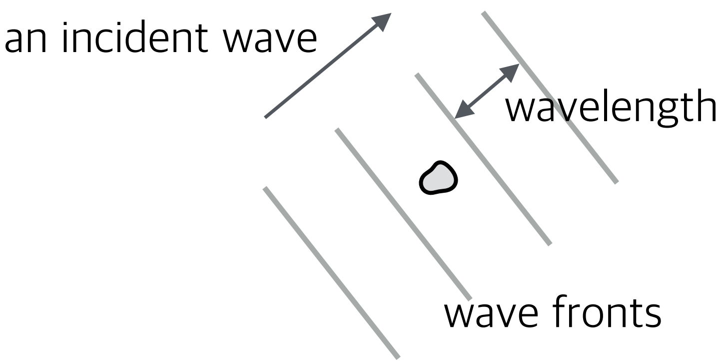
  (Electro-Magnetic, Acoustic, Elastic Waves)

- How? : use sub-wavelength resonators



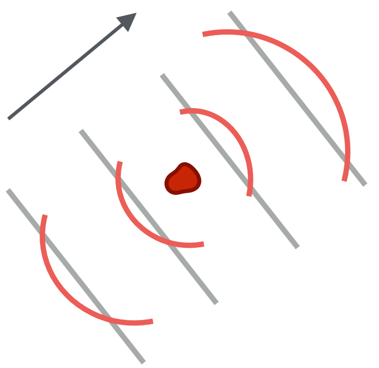
<u>A Sub-wavelength Resonance</u>:

a resonance of a structure whose size is small compared to wavelength



a small ordinary scatterer

→ a negligible scattering



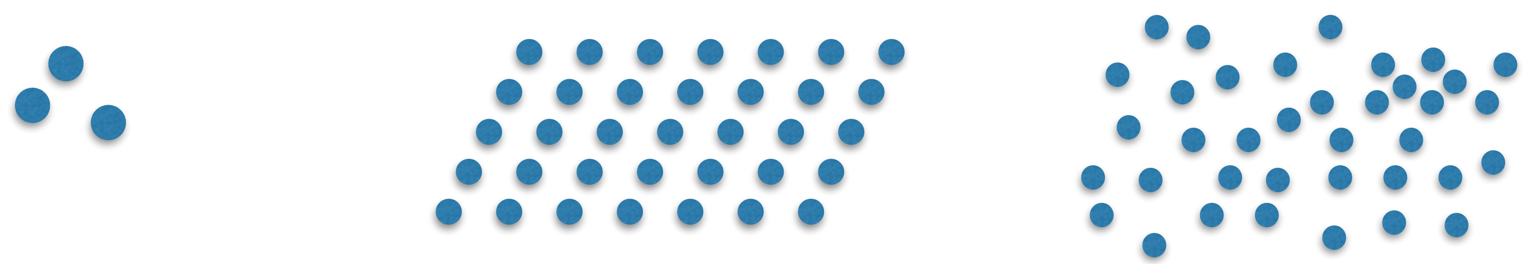
a sub-wavelength resonator

→ a strong scattering

- Examples of meta-materials : negative refraction, cloaking, super-resolution, topological waveguides and so on⋯

Acoustic Metamaterials

- Depending on the geometry of structures, meta-materials can have very different macroscopic behaviors



Ω : a finite system Ω : periodic medium Ω : random media

- **A series of mathematical works on meta-materials**

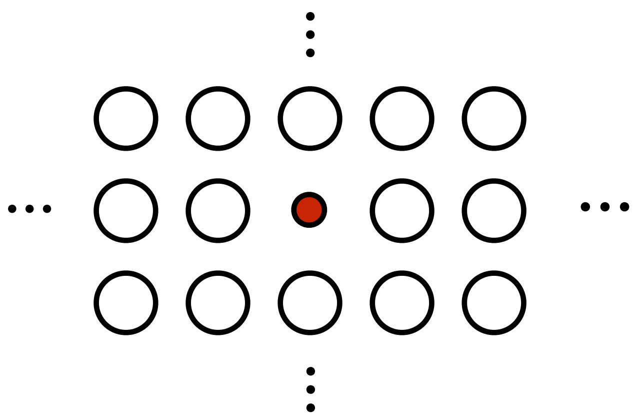
(by Habib Ammari, Bryn Davies, Brian Fitzpatrick, David Gontier, Erik O. Hiltunen, Hyundae Lee, SY, Hai Zhang)

Super-resolution imaging, Negative Refraction, High frequency homogenization,
Dirac Points in Honeycomb Crystals, Topological edge states,
Absorbing Metasurfaces, Exceptional points, Fano Resonances, etc..

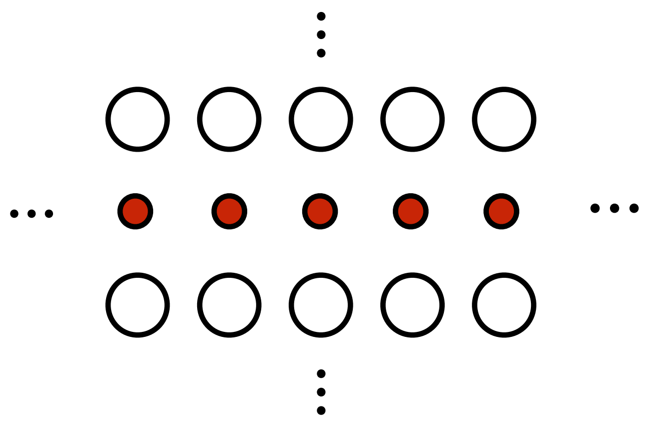
Localization of waves by defects

- In this talk, we focus on the localization of waves by defects

1. Point defect (SIAP 2018)



2. Line defect (JEMS 2022)



(joint works with H. Ammari, B. Fitzpatrick, E.O. Hiltunen)

- Previous works: A. Figotin, A. Krein, V. Goren, M.A. Hoefer, M.I. Weinstein, ⋯
  > not valid for high-contrast coeff PDEs (hence not vaid for metamaterials)
  > mostly qualitative, quantitative results only for low-contrast coeffs PDE

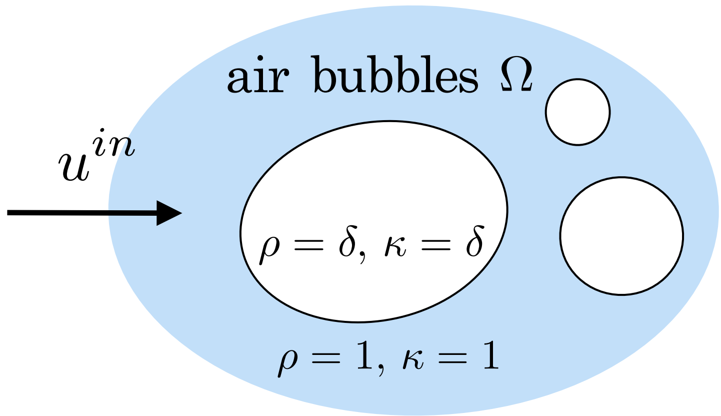
- We develop an integral equation approach to characterize the defect modes motivated by the Fictitious Sources Superposition method (Wilcox et al, PRE 2005)

  > valid for general-contrast coeff PDEs (including metamaterials)
  > quantitative characterization

Acoustic Scattering by Bubbles

- Problem formulation

$$\begin{cases} \nabla \cdot \left(\dfrac{1}{\rho} \nabla u \right) + \omega^2 \dfrac{1}{\kappa} u = 0 & \text{in } \mathbb{R}^d \ (d = 2, 3), \\ \left| \left(\dfrac{d}{d|x|} - i\omega \right) (u - u^{in}) \right| = O\left(|x|^{-(d+1)/2} \right) & \text{as } |x| \to \infty. \end{cases}$$



where

$$\rho = \begin{cases} \delta & \text{in } \Omega, \\ 1 & \text{in } \mathbb{R}^d \setminus \Omega. \end{cases} \qquad \kappa = \begin{cases} \delta & \text{in } \Omega, \\ 1 & \text{in } \mathbb{R}^d \setminus \Omega. \end{cases}$$

mass density distribution — bulk modulus distribution

u : pressure field

ω : frequency

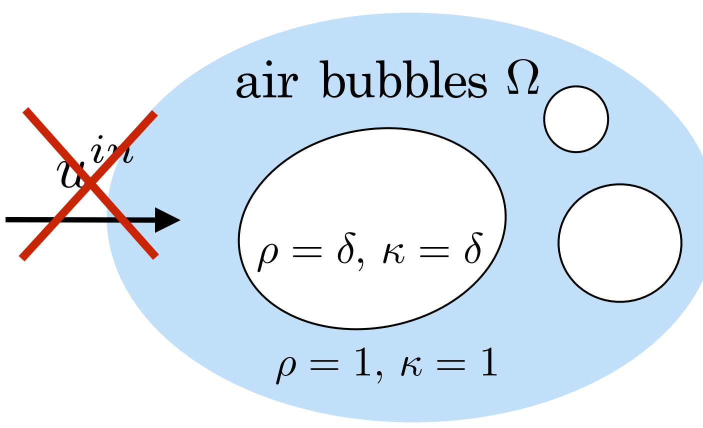
$u(x,t) \sim u(x) e^{-i\omega t}$

- Bubbles satisfy the high-contrast condition $\delta \ll 1$

  (breakdown of the ellipticity when $\delta \to 0$)

Resonances of Acoustic Bubbles

- **Resonances** : suppose that, for some frequency ω^*, there exists a non-trivial solution u^* to the scattering problem with no incident wave



$$\begin{cases} \nabla\cdot\left(\frac{1}{\rho}\nabla u^*\right)+(\omega^*)^2\frac{1}{\kappa}u^*=0 & \text{in } \mathbb{R}^d\ (d=2,3),\\ \left|\left(\frac{d}{d|x|}-i\omega^*\right)u^*\right|=O\big(|x|^{-(d+1)/2}\big) & \text{as } |x|\to\infty. \end{cases}$$

Then ω^* is called the resonance frequency

u^* is called the resonance mode

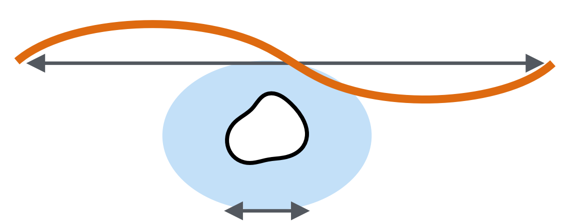
- it can be considered as the eigenvalue problem of the elliptic operator $-\kappa\nabla\cdot\frac{1}{\rho}\nabla$

$$\Big(-\kappa\nabla\cdot\frac{1}{\rho}\nabla\Big)u^*=(\omega^*)^2u^*$$

- sub-wavelength resonances:

a resonance mode u^* with a small resonance frequency ω^*

small frequency $\omega\ll 1$
or
large wavelength $\lambda\gg 1$



Acoustic Scattering by Bubbles (integral equation)

- Integral Equation approach
  - Green's function for (homogenous) Helmholtz equation: $(\Delta + \omega^2)G^\omega = \delta_0$

$$G^\omega(x) = -\frac{i}{4}H_0^{(1)}(\omega|x-y|) \quad \text{(2D)} \qquad G^\omega(x) = -\frac{e^{i\omega|x|}}{4\pi|x|} \quad \text{(3D)}$$

  - Single layer potential

$$\mathcal{S}_\Omega^\omega[\varphi] := \int_{\partial\Omega} G^\omega(x-y)\varphi(y)d\sigma(y) \quad \text{for } \varphi \in L^2(\partial\Omega)$$

  - The scattered field can be represented using the single layer potentials

$$u = \begin{cases} u^{in} + \mathcal{S}_\Omega^\omega[\psi] & \text{in } \mathbb{R}^d \setminus \overline{\Omega} \\ \mathcal{S}_\Omega^\omega[\varphi] & \text{in } \Omega \end{cases}$$

  - How can we determine the source density functions (ψ, φ)?

Acoustic Scattering by Bubbles (integral equation)

Integral Equation

- The source density functions satisfy

$$\mathcal{A}^{\delta}_{\Omega}(\omega)\begin{pmatrix}\varphi\\ \psi\end{pmatrix} := \begin{pmatrix}\mathcal{S}^{\omega}_{\Omega} & -\mathcal{S}^{\omega}_{\Omega}\\ \partial_\nu \mathcal{S}^{\omega}_{\Omega}|_- & -\delta\partial_\nu \mathcal{S}^{\omega}_{\Omega}|_+\end{pmatrix}\begin{pmatrix}\varphi\\ \psi\end{pmatrix} = \begin{pmatrix}u^{in}|_{\partial\Omega}\\ \partial_\nu u^{in}|_{\partial\Omega}\end{pmatrix}$$

Resonances (integral equation formulation)

- We look for a frequency ω^* such that there exists a nontrivial solution (φ^*, ψ^*) to

$$\mathcal{A}^{\delta}_{\Omega}(\omega^*)\begin{pmatrix}\varphi^*\\ \psi^*\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

- ω^* is the resonance frequency
- (φ^*, ψ^*) gives the resonance mode

$$u^* = \begin{cases} 0 + \mathcal{S}^{\omega^*}_{\Omega}[\psi^*] & \text{in } \mathbb{R}^d \setminus \overline{\Omega} \\ \mathcal{S}^{\omega^*}_{\Omega}[\varphi^*] & \text{in } \Omega \end{cases}$$

Sub-wavelength Resonances of Bubbles

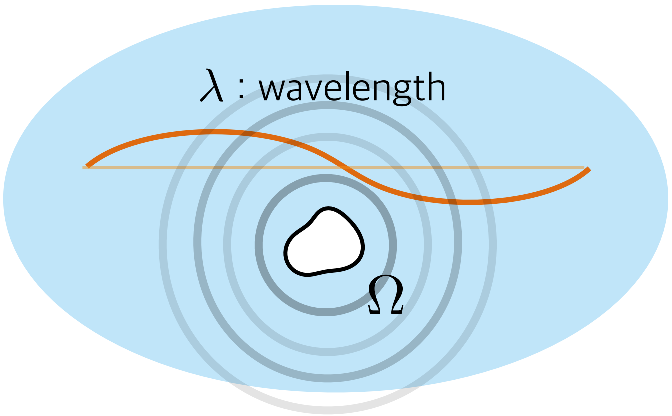
- It can be shown that a bubble can have a "sub-wavelength" resonance whose resonance frequency is small when the high contrast parameter δ is small

- a single bubble case

  - sub-wavelength resonance frequency ω_δ

$$\omega_\delta = \sqrt{\frac{\mathrm{Cap}_\Omega}{|\Omega|}}\delta^{\frac{1}{2}} - i\frac{\mathrm{Cap}_\Omega^2}{8\pi|\Omega|}\delta + O(\delta^{\frac{3}{2}}) \quad \text{as } \delta \to 0$$

Cap_Ω : capacity associated to the domain Ω



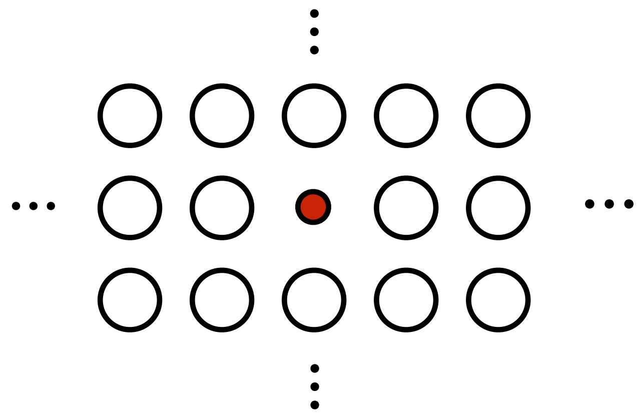
  - A single bubble is a strong monopole scatterer

$$(u - u^{in})(x) \approx \frac{\mathrm{Cap}_\Omega}{1 - \left(\frac{\omega_M}{\omega}\right) + i\gamma} u^{in}(0) G^\omega(x), \quad \text{for large } |x|.$$

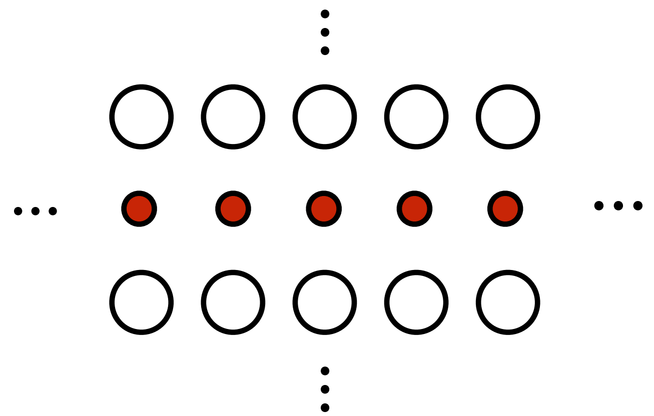
$\longrightarrow$ "a bubble is a good sub-wavelength resonator"

Subwavelength localized modes for acoustic waves

1. Point defect (SIAP 2018)



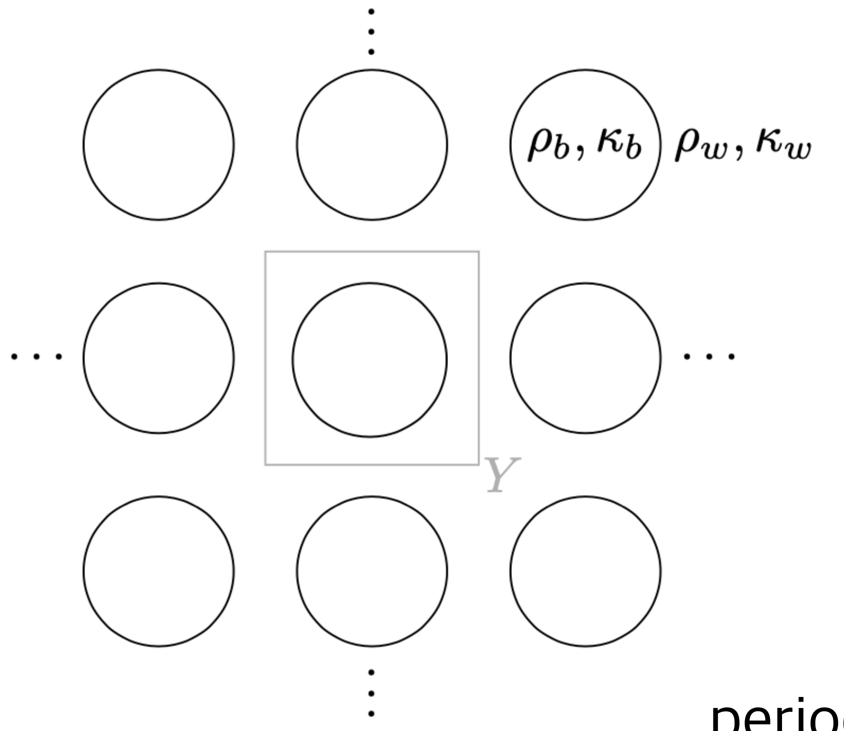
2. Line defect (JEMS 2022)



(joint works with H. Ammari, B. Fitzpatrick, E.O. Hiltunen)

Localized Modes by Defects

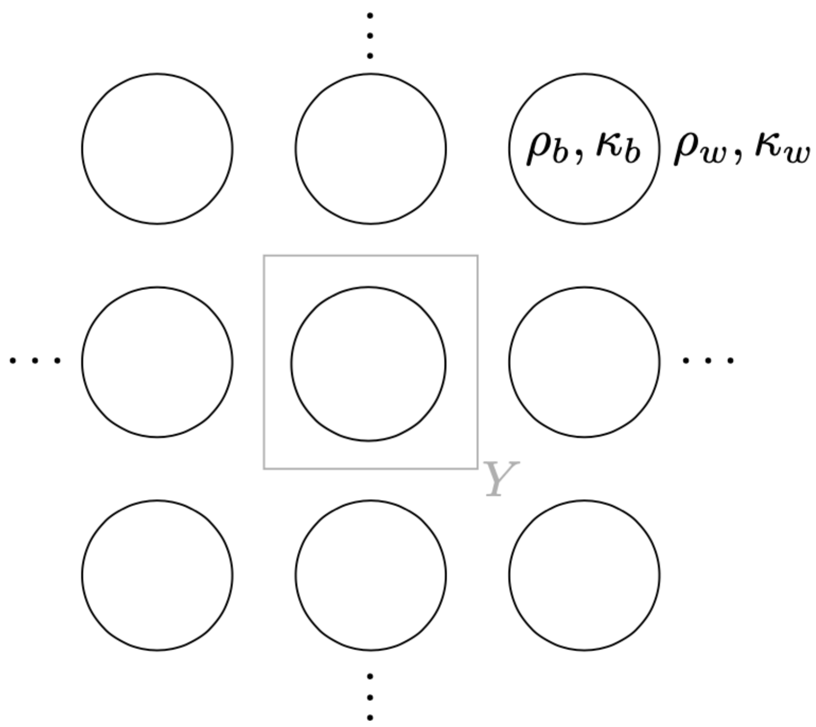
- Before considering the defects, let us briefly review wave propagation in the perfectly periodic structures



periodic (no defect)

Localized Modes by Defects
- Point Defect Case

- a perfectly periodic structure



Bloch waves

$$v(x) = w(x)e^{i\alpha\cdot x}, \quad w : \text{periodic } in \text{ Y}$$
$$\alpha : \text{Bloch momentum}$$

$$\begin{cases} \nabla \cdot \dfrac{1}{\rho_w}\nabla v + \dfrac{\omega^2}{\kappa_w} v = 0 \quad \text{in} \quad Y\backslash\overline{D}, \\ \nabla \cdot \dfrac{1}{\rho_b}\nabla v + \dfrac{\omega^2}{\kappa_b} v = 0 \quad \text{in} \quad D, \\ v|_+ - v|_- = 0 \quad \text{on} \quad \partial D, \\ \dfrac{1}{\rho_w}\dfrac{\partial v}{\partial \nu}\Big|_+ - \dfrac{1}{\rho_b}\dfrac{\partial v}{\partial \nu}\Big|_- = 0 \quad \text{on} \quad \partial D, \\ e^{-i\alpha\cdot x} v \ \text{ is periodic.} \end{cases}$$

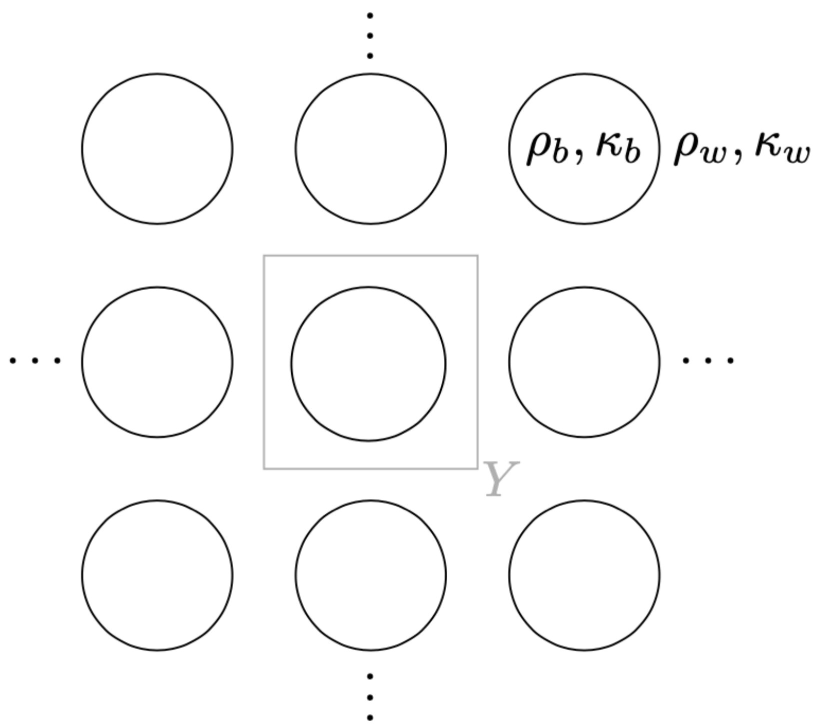
- Integral equation formulation

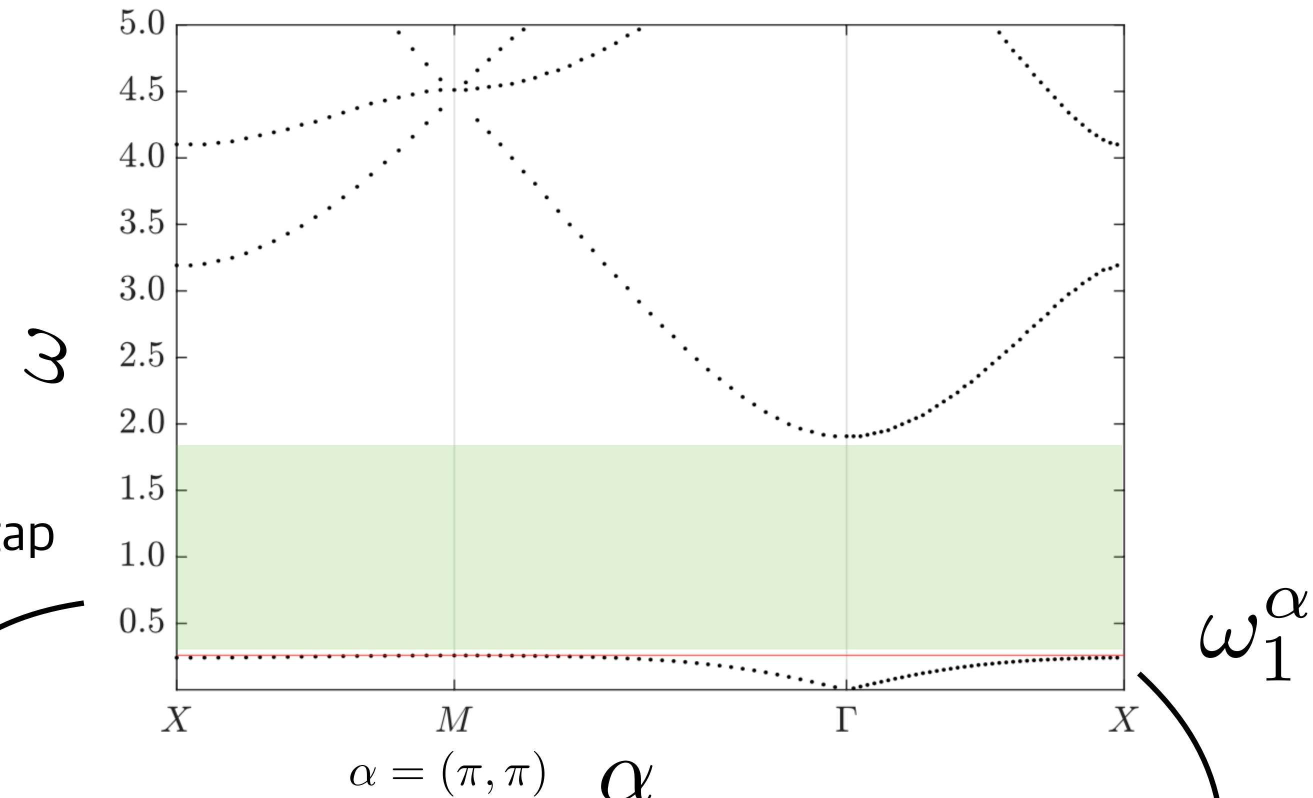
$$v = \begin{cases} \mathcal{S}_D^{\alpha,\omega}[\psi^\alpha] & \text{in } Y \setminus \overline{D} \\ \mathcal{S}_D^{\omega}[\varphi^\alpha] & \text{in } D \end{cases} \qquad \mathcal{A}^\alpha(\omega)\begin{pmatrix}\varphi^\alpha \\ \psi^\alpha\end{pmatrix} := \begin{pmatrix} \mathcal{S}_D^\omega & -\mathcal{S}_D^{\alpha,\omega} \\ \partial_\nu \mathcal{S}_D^\omega|_- & -\delta\partial_\nu \mathcal{S}_D^{\alpha,\omega}|_+ \end{pmatrix}\begin{pmatrix}\varphi^\alpha \\ \psi^\alpha\end{pmatrix} = \begin{pmatrix}0 \\ 0\end{pmatrix}$$

Here, $\mathcal{S}_D^{\alpha,\omega}$ is the quasi-periodic single layer potential with the kernel $G^{\alpha,\omega}(x) = \sum_{n\in\mathbb{R}^2} G^\omega(x-n)e^{in\cdot\alpha}$

Localized Modes by Defects
- Point Defect Case

- We begin with a perfectly periodic crystal



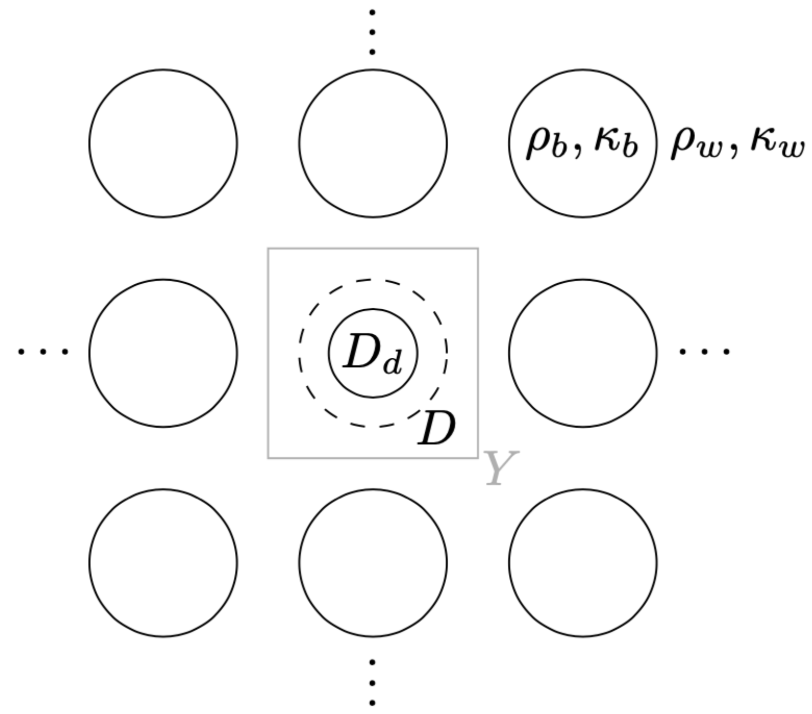


- We can prove that a band-gap opens in the sub-wavelength regime (in the band-gap, waves cannot propagate)

- the first sub-wavelength band (waves can propagate)

Localized Modes by Defects
- Point Defect Case

- We next introduce a point defect !

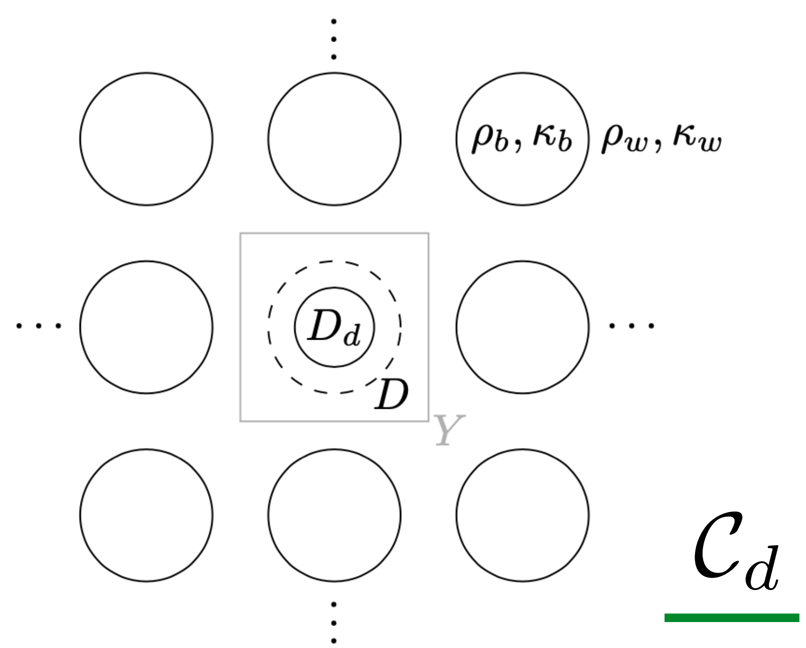


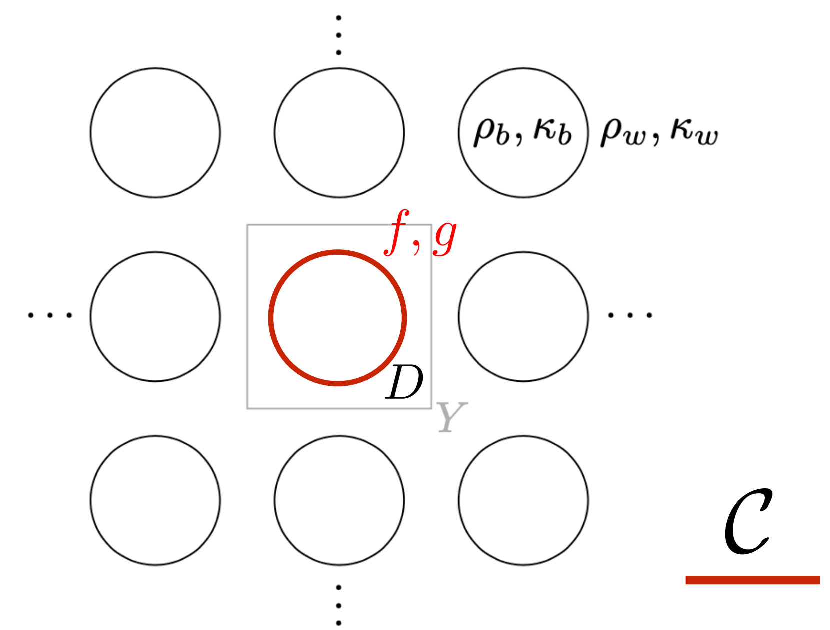
- No periodicity → the Bloch wave analysis is not directly applicable

- We apply an integral equation approach to characterize the defect modes motivated by the Fictitious Sources Superposition method (Wilcox et al, PRE 2005)

Localized Modes by Defects
- Point Defect Case

STEP 1 Fictitous Sources Characterization





$$\begin{cases} \nabla \cdot \frac{1}{\rho_w} \nabla u + \frac{\omega^2}{\kappa_w} u = 0 & \text{in} \quad \mathbb{R}^2 \backslash \mathcal{C}_d, \\ \nabla \cdot \frac{1}{\rho_b} \nabla u + \frac{\omega^2}{\kappa_b} u = 0 & \text{in} \quad \mathcal{C}_d, \\ u|_+ - u|_- = \boxed{0} & \text{on} \quad \partial \mathcal{C}_d, \\ \frac{1}{\rho_w} \frac{\partial u}{\partial \nu}\Big|_+ - \frac{1}{\rho_b} \frac{\partial u}{\partial \nu}\Big|_- = \boxed{0} & \text{on} \quad \partial \mathcal{C}_d. \end{cases}$$

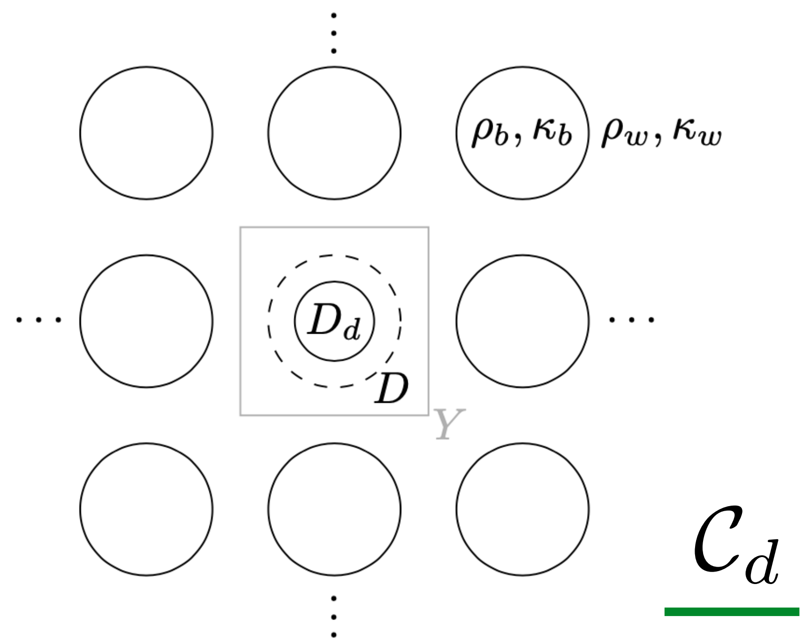
- geometry : non-periodic
- interface conditions: standard

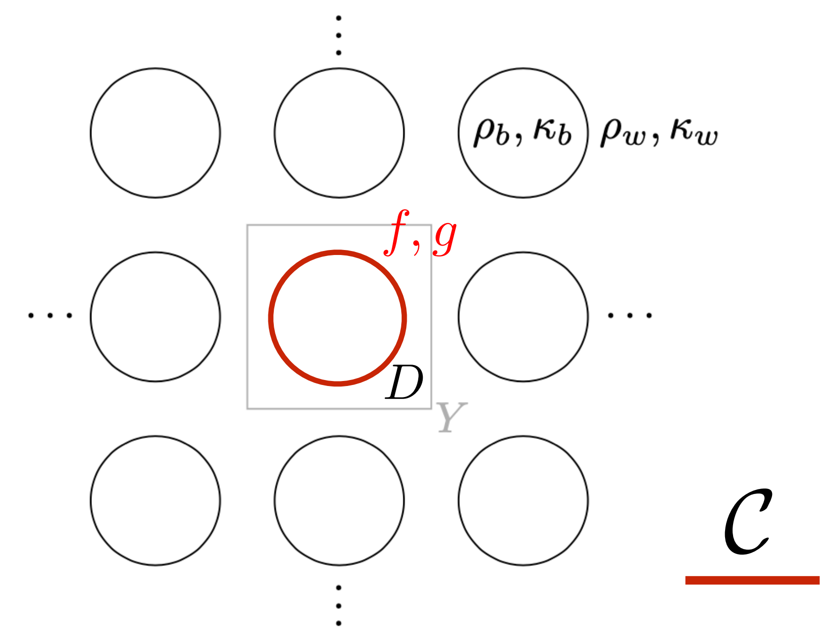
$$\begin{cases} \nabla \cdot \frac{1}{\rho_w} \nabla \tilde{u} + \frac{\omega^2}{\kappa_w} \tilde{u} = 0 & \text{in} \quad \mathbb{R}^2 \setminus \mathcal{C}, \\ \nabla \cdot \frac{1}{\rho_b} \nabla \tilde{u} + \frac{\omega^2}{\kappa_b} \tilde{u} = 0 & \text{in} \quad \mathcal{C}, \\ \tilde{u}|_+ - \tilde{u}|_- = \boxed{f \delta_{m,(0,0)}} & \text{on} \quad \partial D + m, \ m \in \mathbb{Z}^2, \\ \frac{1}{\rho_w} \frac{\partial \tilde{u}}{\partial \nu}\Big|_+ - \frac{1}{\rho_b} \frac{\partial \tilde{u}}{\partial \nu}\Big|_- = \boxed{g \delta_{m,(0,0)}} & \text{on} \quad \partial D + m, \ m \in \mathbb{Z}^2. \end{cases}$$

- geometry : periodic
- interface conditions: fictitious sources at central bubble

Localized Modes by Defects
- Point Defect Case

‣ STEP 1 Fictitous Sources Characterization





$$u = \begin{cases} H + \mathcal{S}^{\omega}_{D_d}[\psi_d] & \text{in } Y \setminus \overline{D_d} \\ \mathcal{S}^{\omega}_{D_d}[\varphi_d] & \text{in } D_d \end{cases}$$

$$(\Delta + \omega^2)H = 0 \text{ in } Y$$

where

$$\mathcal{A}_{D_d}\begin{pmatrix}\varphi_d \\ \psi_d\end{pmatrix} = \begin{pmatrix} H|_{\partial D_d} \\ \partial_\nu H|_{\partial D_d}\end{pmatrix}$$

$$\mathcal{A}_{D_d} := \begin{pmatrix} \mathcal{S}^{\omega}_{D_d} & -\mathcal{S}^{\omega}_{D_d} \\ \partial_\nu \mathcal{S}^{\omega}_{D_d}|_- & -\delta\partial_\nu \mathcal{S}^{\omega}_{D_d}|_+ \end{pmatrix}$$

$$\tilde{u} = \begin{cases} H + \mathcal{S}^{\omega}_{D}[\psi] & \text{in } Y \setminus \overline{D} \\ \mathcal{S}^{\omega}_{D}[\varphi] & \text{in } D \end{cases}$$

$$(\Delta + \omega^2)H = 0 \text{ in } Y$$

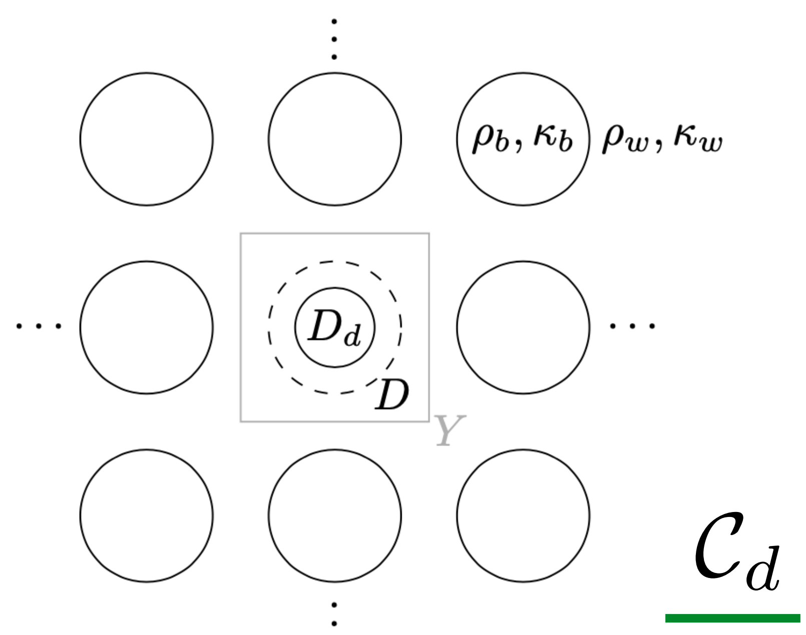
where

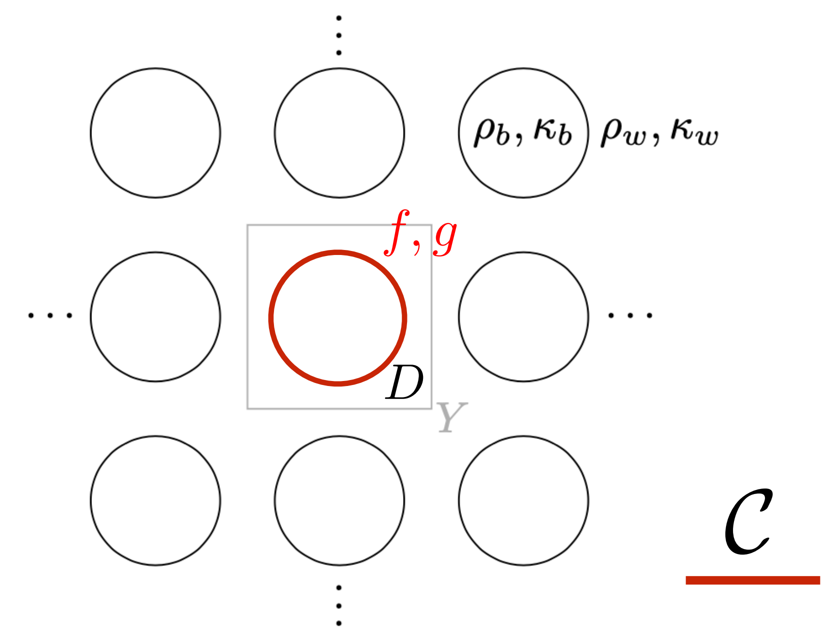
$$\mathcal{A}_{D}\begin{pmatrix}\varphi \\ \psi\end{pmatrix} = \begin{pmatrix} H|_{\partial D} - f \\ \partial_\nu H|_{\partial D} - g\end{pmatrix}$$

$$\mathcal{A}_{D} := \begin{pmatrix} \mathcal{S}^{\omega}_{D} & -\mathcal{S}^{\omega}_{D} \\ \partial_\nu \mathcal{S}^{\omega}_{D}|_- & -\delta\partial_\nu \mathcal{S}^{\omega}_{D}|_+ \end{pmatrix}$$

Localized Modes by Defects
- Point Defect Case

STEP 1 Fictitous Sources Characterization





$$u = \begin{cases} H + \mathcal{S}_{D_d}^\omega[\psi_d] & \text{in } Y \setminus \overline{D_d} \\ \mathcal{S}_{D_d}^\omega[\varphi_d] & \text{in } D_d \end{cases}$$

$$(\Delta + \omega^2)H = 0 \text{ in } Y$$

$$\tilde{u} = \begin{cases} H + \mathcal{S}_{D}^\omega[\psi] & \text{in } Y \setminus \overline{D} \\ \mathcal{S}_{D}^\omega[\varphi] & \text{in } D \end{cases}$$

$$(\Delta + \omega^2)H = 0 \text{ in } Y$$

where

$$\mathcal{A}_{D_d}\begin{pmatrix}\varphi_d \\ \psi_d\end{pmatrix} = \begin{pmatrix} H|_{\partial D_d} \\ \partial_\nu H|_{\partial D_d}\end{pmatrix}$$

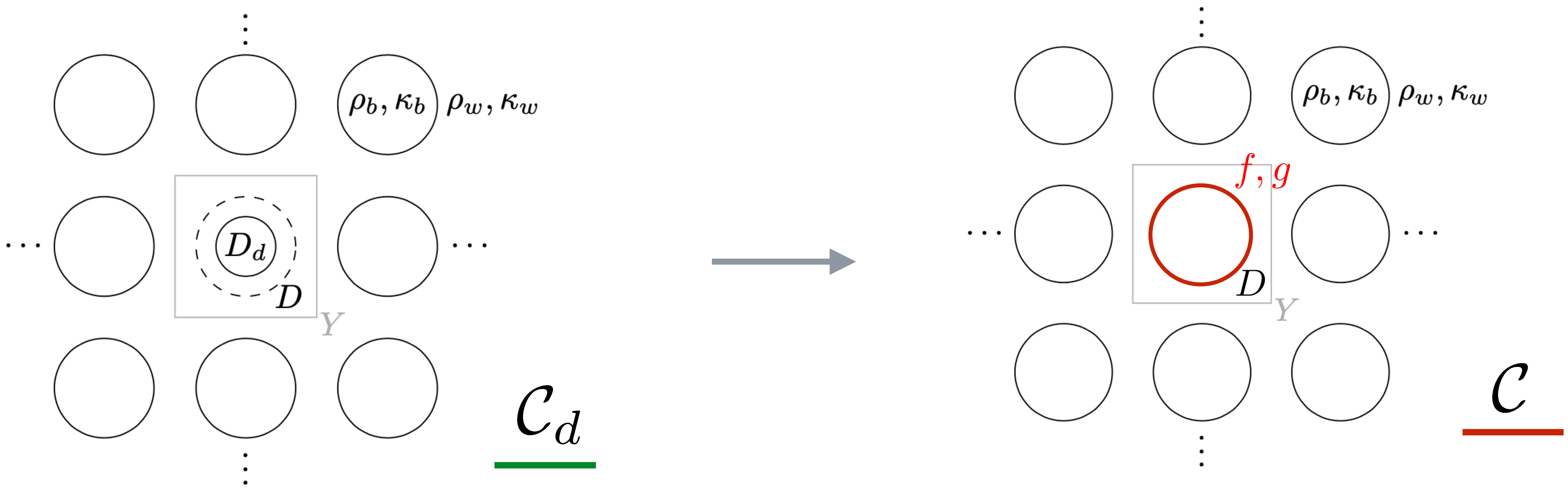
$$\mathcal{A}_{D_d} := \begin{pmatrix} \mathcal{S}_{D_d}^\omega & -\mathcal{S}_{D_d}^\omega \\ \partial_\nu \mathcal{S}_{D_d}^\omega|_- & -\delta\partial_\nu \mathcal{S}_{D_d}^\omega|_+ \end{pmatrix}$$

In order to mimic the defect, we want to have
the same scattered waves $\mathcal{S}_{D_d}^\omega[\varphi_d] \equiv \mathcal{S}_D^\omega[\varphi]$ $\mathcal{S}_{D_d}^\omega[\psi_d] \equiv \mathcal{S}_D^\omega[\psi]$
for the same incident waves H

$$\mathcal{A}_{D} := \begin{pmatrix} \mathcal{S}_{D}^\omega & -\mathcal{S}_{D}^\omega \\ \partial_\nu \mathcal{S}_{D}^\omega|_- & -\delta\partial_\nu \mathcal{S}_{D}^\omega|_+ \end{pmatrix}$$

Localized Modes by Defects
- Point Defect Case

STEP 1 Fictitous Sources Characterization



- It is possible to construct the operator $\mathcal{A}_D^{fict}$ on the unperturbed bubble ∂D which mimics the behavior of the defect bubble operator $\mathcal{A}_{D_d}$

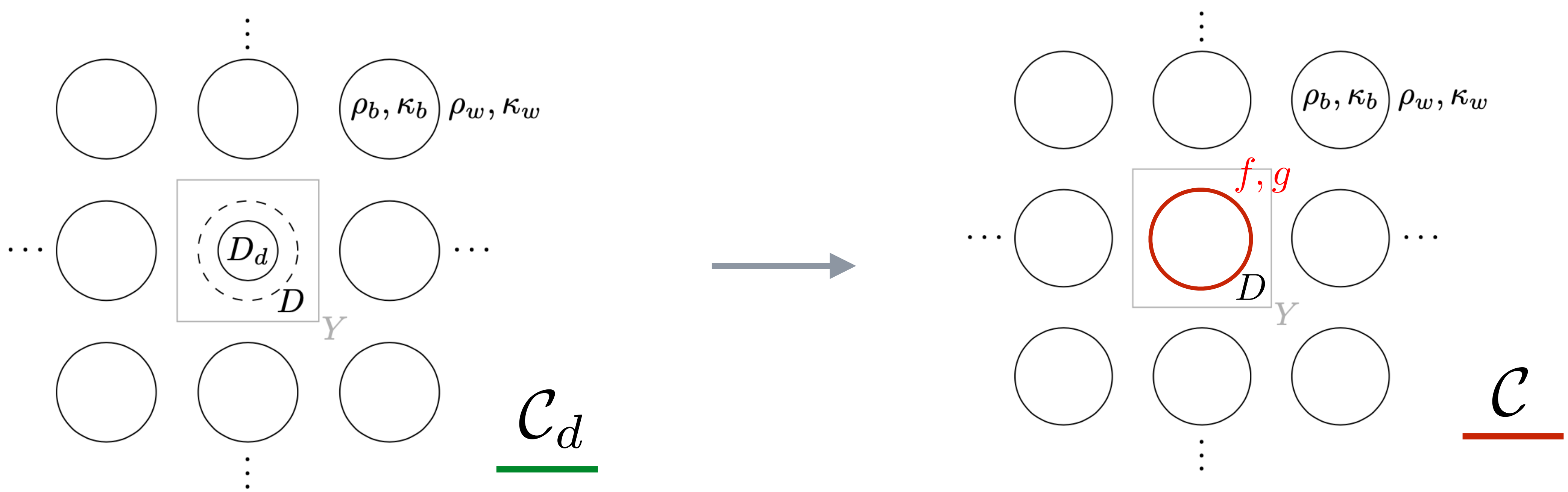
$$\mathcal{A}_{D_d}\begin{pmatrix}\varphi_d\\ \psi_d\end{pmatrix} = \begin{pmatrix}H|_{\partial D_d}\\ \partial_\nu H|_{\partial D_d}\end{pmatrix} \qquad \mathcal{A}_D^{fict}\begin{pmatrix}\varphi\\ \psi\end{pmatrix} = \begin{pmatrix}H|_{\partial D}\\ \partial_\nu H|_{\partial D}\end{pmatrix} \longrightarrow \begin{matrix}\mathcal{S}_{D_d}^{\omega}[\psi_d] \equiv \mathcal{S}_D^{\omega}[\psi]\\ \mathcal{S}_{D_d}^{\omega}[\varphi_d] \equiv \mathcal{S}_D^{\omega}[\varphi]\end{matrix} \text{ in appropriate regions}$$

the same incident waves

the same scattered waves

Localized Modes by Defects
- Point Defect Case

STEP 1 Fictitous Sources Characterization



$$\mathcal{A}_D \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} H|_{\partial D} - f \\ \partial_\nu H|_{\partial D} - g \end{pmatrix}$$

and

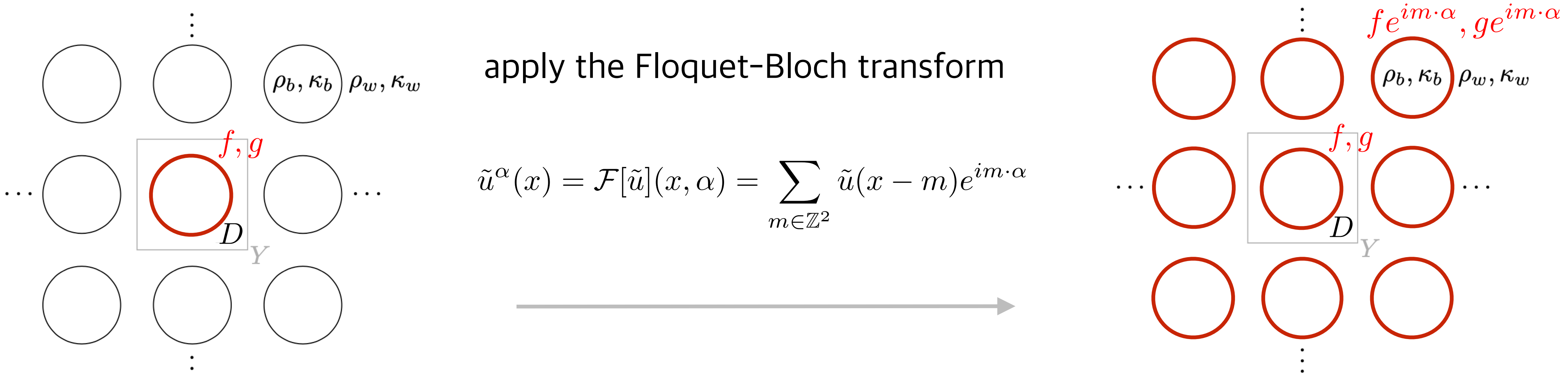
$$\mathcal{A}_D^{fict} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} H|_{\partial D} \\ \partial_\nu H|_{\partial D} \end{pmatrix}$$

So the fictitious sources (f, g) should satisfy

$$(\mathcal{A}_D^{fict} - \mathcal{A}_D) \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$$

Localized Modes by Defects
- Point Defect Case

STEP 2 Quasi-periodization by Floquet-Bloch transform



$$\tilde{u}^\alpha(x) = \mathcal{F}[\tilde{u}](x,\alpha) = \sum_{m\in\mathbb{Z}^2} \tilde{u}(x-m)e^{im\cdot\alpha}$$

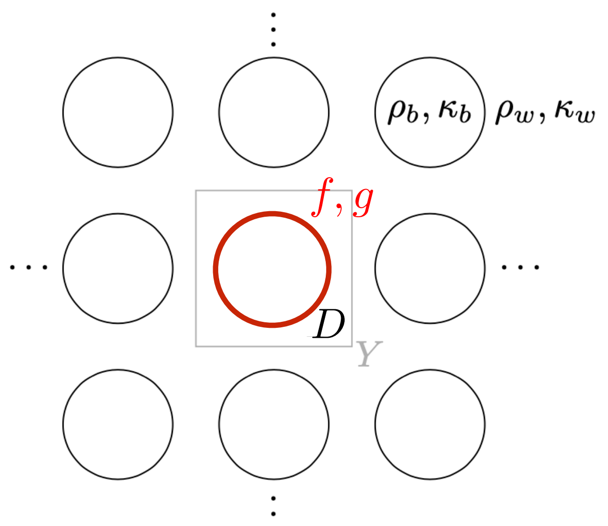
$$\begin{cases} \nabla\cdot\frac{1}{\rho_w}\nabla\tilde{u} + \frac{\omega^2}{\kappa_w}\tilde{u} = 0 \quad \text{in} \quad \mathbb{R}^2\setminus\mathcal{C}, \\ \nabla\cdot\frac{1}{\rho_b}\nabla\tilde{u} + \frac{\omega^2}{\kappa_b}\tilde{u} = 0 \quad \text{in} \quad \mathcal{C}, \\ \tilde{u}|_+ - \tilde{u}|_- = f\delta_{m,(0,0)} \quad \text{on} \quad \partial D + m,\ m\in\mathbb{Z}^2, \\ \frac{1}{\rho_w}\frac{\partial\tilde{u}}{\partial\nu}\Big|_+ - \frac{1}{\rho_b}\frac{\partial\tilde{u}}{\partial\nu}\Big|_- = g\delta_{m,(0,0)} \quad \text{on} \quad \partial D + m,\ m\in\mathbb{Z}^2. \end{cases}$$

$$\begin{cases} \nabla\cdot\frac{1}{\rho_w}\nabla u^\alpha + \frac{\omega^2}{\kappa_w}u^\alpha = 0 \quad \text{in} \quad Y\setminus\overline{D}, \\ \nabla\cdot\frac{1}{\rho_b}\nabla u^\alpha + \frac{\omega^2}{\kappa_b}u^\alpha = 0 \quad \text{in} \quad D, \\ u^\alpha|_+ - u^\alpha|_- = f \quad \text{on} \quad \partial D, \\ \frac{1}{\rho_w}\frac{\partial u^\alpha}{\partial\nu}\Big|_+ - \frac{1}{\rho_b}\frac{\partial u^\alpha}{\partial\nu}\Big|_- = g \quad \text{on} \quad \partial D, \\ e^{-i\alpha\cdot x}u^\alpha \text{ is periodic.} \end{cases}$$

→ a quasi-periodic problem!

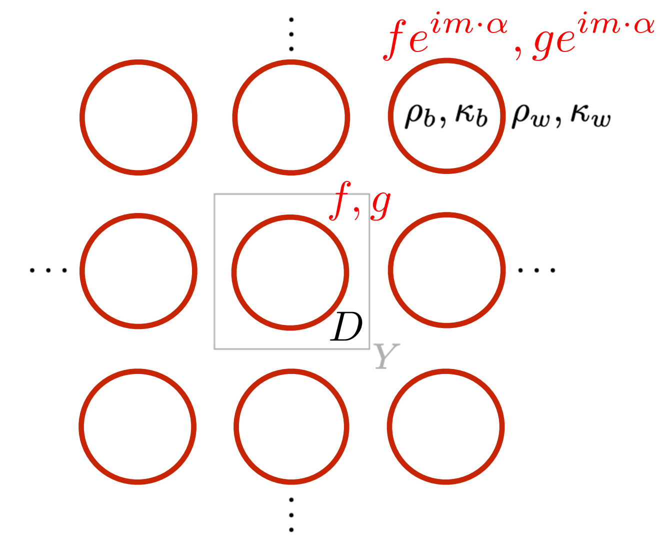
Localized Modes by Defects
- Point Defect Case

STEP 2 Quasi-periodization by Floquet-Bloch transform



apply the Floquet-Bloch transform

$$\tilde{u}^\alpha(x) = \mathcal{F}[\tilde{u}](x,\alpha) = \sum_{m\in\mathbb{Z}^2} \tilde{u}(x-m)e^{im\cdot\alpha}$$



$$\tilde{u} = \begin{cases} H + \mathcal{S}_D^\omega[\psi] & \text{in } Y\setminus\overline{D} \\ \mathcal{S}_D^\omega[\varphi] & \text{in } D \end{cases}$$

$$\tilde{u}^\alpha = \begin{cases} \mathcal{S}_D^{\alpha,\omega}[\psi^\alpha] & \text{in } Y\setminus\overline{D} \\ \mathcal{S}_D^\omega[\varphi^\alpha] & \text{in } D \end{cases}$$

$$\begin{pmatrix}\varphi \\ \psi\end{pmatrix} = \frac{1}{(2\pi)^2}\int_{BZ} (\mathcal{A}^\alpha)^{-1}\begin{pmatrix} f \\ g\end{pmatrix} d\alpha$$

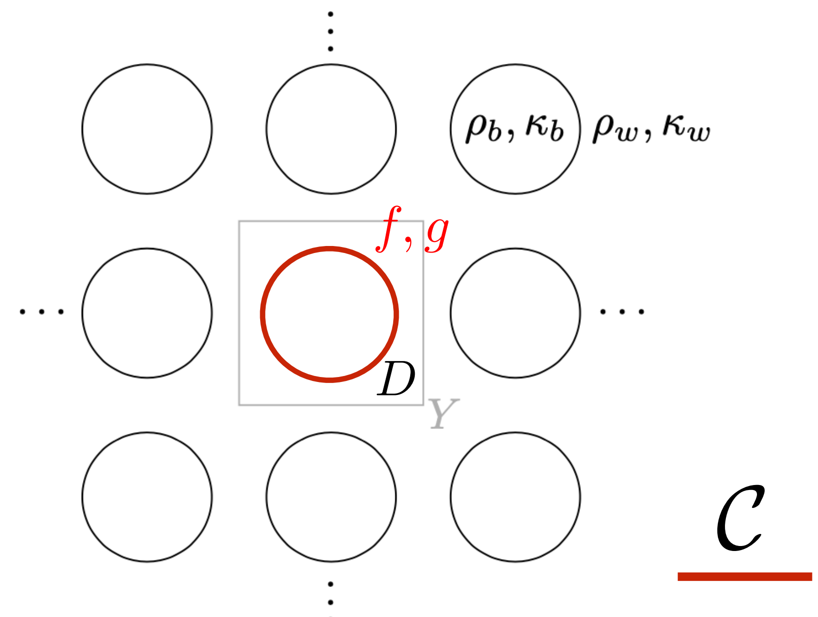
$$\begin{pmatrix}\varphi^\alpha \\ \psi^\alpha\end{pmatrix} = (\mathcal{A}^\alpha)^{-1}\begin{pmatrix} -f \\ -g\end{pmatrix}$$

$$\mathcal{F}^{-1}[\tilde{u}^\alpha](x) = \frac{1}{(2\pi)^2}\int_{BZ} \tilde{u}^\alpha(x,\alpha)d\alpha$$

apply the inverse transform

Localized Modes by Defects
- Point Defect Case

▸ STEP 3 Integral equation for fictitious sources



STEP 1

$$(\mathcal{A}_D^{fict} - \mathcal{A}_D)\begin{pmatrix}\varphi\\ \psi\end{pmatrix} = \begin{pmatrix}f\\ g\end{pmatrix}$$

STEP 2

$$\begin{pmatrix}\varphi\\ \psi\end{pmatrix} = \frac{1}{(2\pi)^2}\int_{BZ}(\mathcal{A}^\alpha)^{-1}\begin{pmatrix}f\\ g\end{pmatrix} d\alpha$$

→

integral equation for fictitious sources

$$\left(I + (\mathcal{A}_D^{fict} - \mathcal{A}_D)\frac{1}{(2\pi)^2}\int_{BZ}(\mathcal{A}^\alpha)^{-1} d\alpha\right)\begin{pmatrix}f\\ g\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

Gohberg-Sigal theory can be applied

Localized Modes by Defects

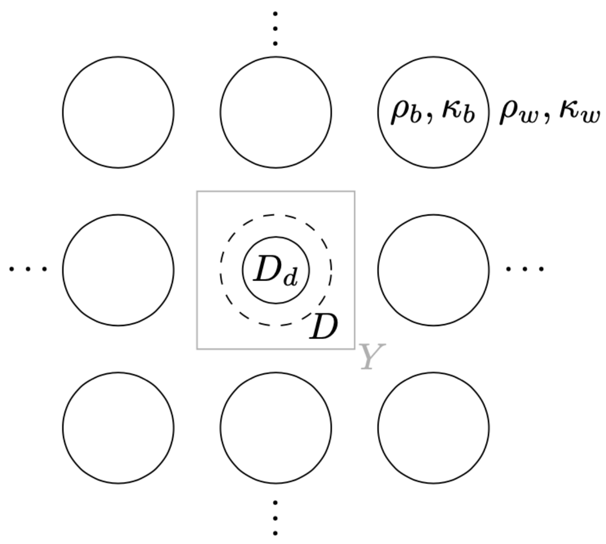
- Point Defect Case

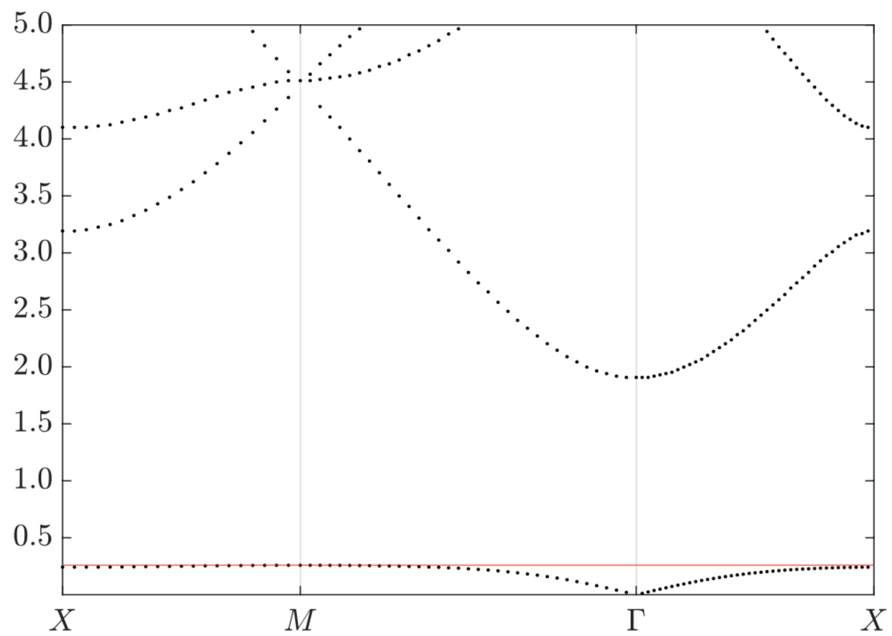
Main results 1 (H. Ammari, B. Fitzpatrick, E. Hiltunen & S.Y., SIAP 2018)

- the existence of the the defect localized mode:
  1. dilute case: the defect should be smaller
  2. non-dilute case: the defect should be larger
- the quantitative asymptotic formula:

$$\omega^d - \omega^* \sim \exp\left(-c_R \frac{4\pi^2 R^3 \omega^*}{\delta(R_d - R)}\right) \qquad \text{as } \delta \to 0$$

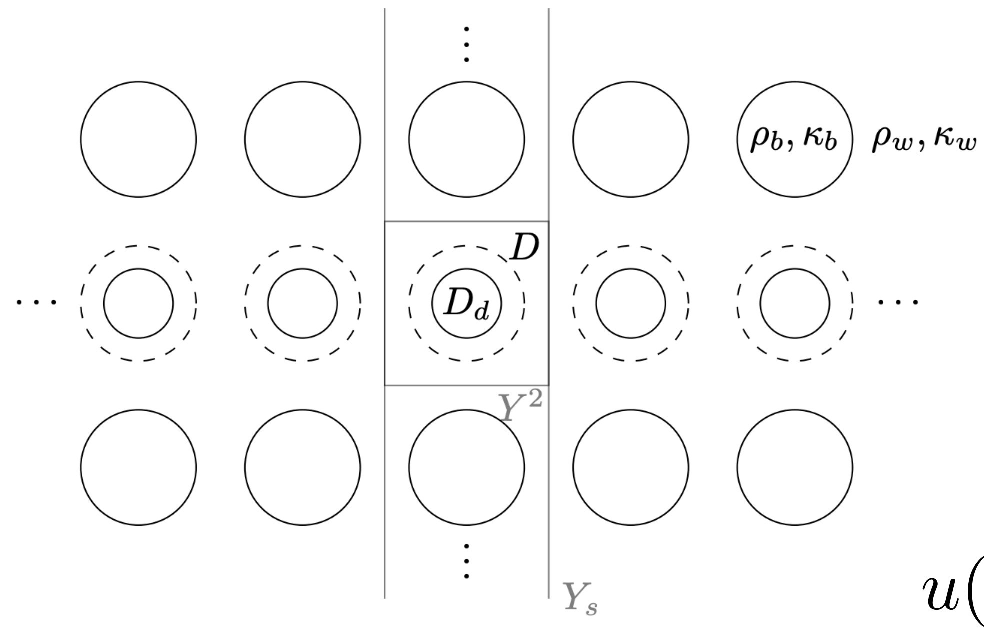
where $\omega^* := \max_\alpha \omega_1^\alpha$





Localized Modes by Defects

- Line Defect Case



$$u(x) = v(x)e^{i\alpha_1 x_1}$$

- periodic in the x_1-direction → the Bloch wave decomposition w.r.t. α_1
  → reduced to the problem on the single strip Y_s

- On the single strip Y_s, the defect bubble D_d is a point defect
  → the fictitious sources method can be applied

$$\left(I + \left(\frac{1}{2\pi}\int_{Y^*} \mathcal{A}^{\alpha}(\omega,\delta)^{-1}\,\mathrm{d}\alpha_2\right)\left(\mathcal{A}_D^{\varepsilon}(\omega,\delta) - \mathcal{A}_D(\omega,\delta)\right)\right)\begin{pmatrix}\phi\\ \psi\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

- the defect mode propagates in the x_1-direction and localized in the x_2-direction → the guided modes along the line defect

Localized Modes by Defects

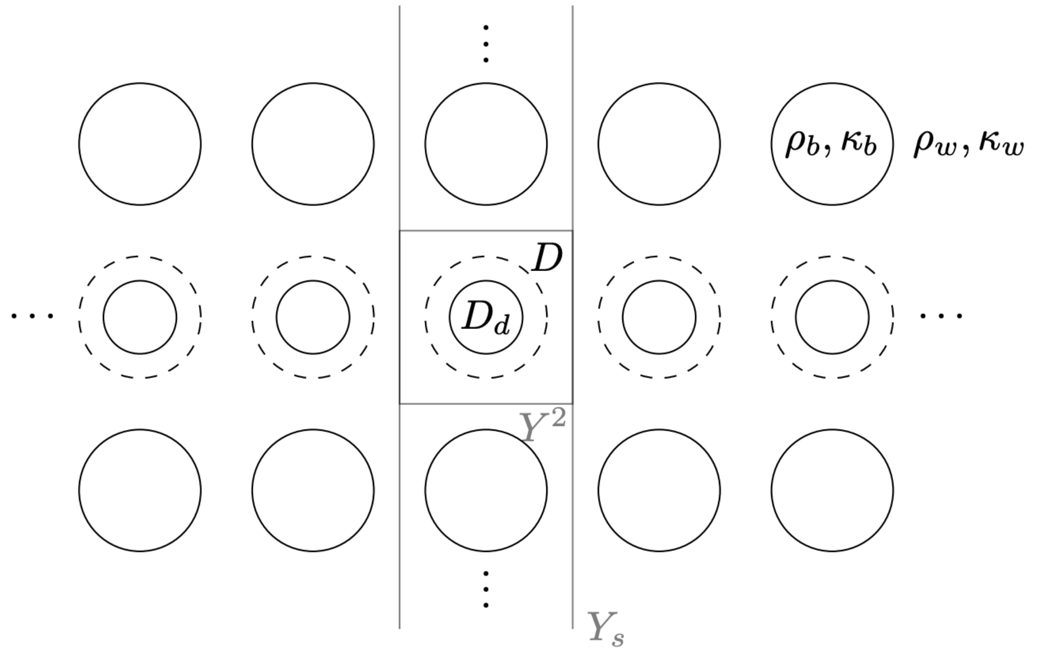
- Line Defect Case

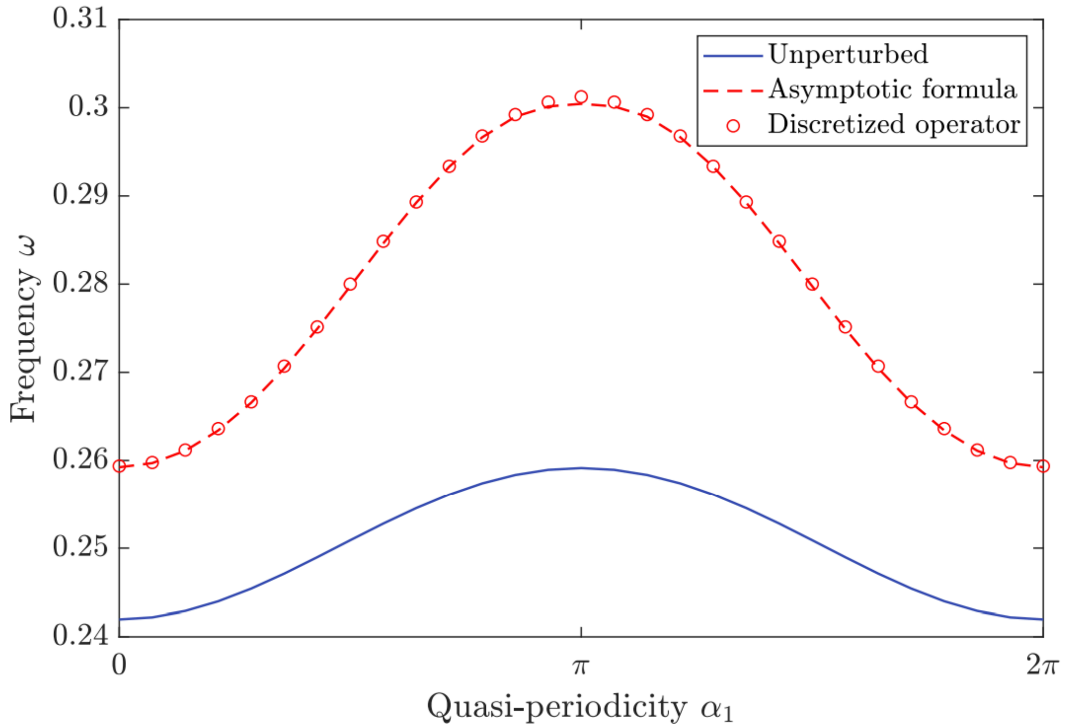
Main result 2 (H. Ammari, E. Hiltunen & S.Y., JEMS 2022)

- the existence of the line defect guided mode
- the quantitative asymptotic formula for the dispersion relation:

$$\omega^d(\alpha_1) \sim \omega^*(\alpha_1) + \frac{1}{2C_\delta(\alpha_1)} \left(\frac{\delta(R_d - R)}{\omega^*(\alpha_1) R^3} \right)^2 \quad \text{as } \delta \to 0$$

where $\omega^*(\alpha_1) := \omega_1^{(\alpha_1, \pi)}$





Conclusion

1. We quantitatively characterized the defect modes (point defect & line defect) at deep sub-wavelength scales
2. asymptotic formulas are useful for optimal geometry design of defect modes
3. numerical illustrations validating theoretical results
4. future works: topologically protected guided modes

Thank you