Discrete Fractal Dimensions of the Ranges of Random Walks Associate with Random Conductances

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Outline

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Summary
The Random Conductance Model

- \( \mathbb{Z}^d \) = \( d \)-dimension integer lattice; \( E_d = \{ \text{non-oriented nearest neighbor bonds} \} \)
- **Environment**: for a given distribution \( \mathbb{Q} \) on \([0, \infty)\),

\[
\mu_e \sim_{i.i.d.} \mathbb{Q}, \quad \text{for all } e \in E_d;
\]

- Given a realization \( \omega = \{ \mu_e : e \in E_d \} \), two random walks:
  1. Variable speed random walk (VSRW), \((X_t)\), waits at \( x \) for an exponential time with mean \( 1/\mu_x \);
  2. Constant speed random walk (CSRW), \((Y_t)\), waits at \( x \) for an exponential time with mean 1;

and then jumps to a neighboring site \( y \) with probability

\[
P_{xy}(\omega) = \frac{\mu_{xy}}{\mu_x} \quad \text{where} \quad \mu_x = \sum_{y \sim x} \mu_{xy}.
\]
Transition Probabilities

![Diagram of transition probabilities with labels 2/10, 1/10, 3/10, 4/10 and corresponding nodes 2, 1, 3, 4.]
Examples

Eg 1:

- \( Q = \delta_{\{1\}} \), then \( \mu_e \) are constantly 1, and \( Y_t \) is just the usual nearest neighbor random walk
- Functional CLT (FCLT):

\[
\frac{Y_{nt}}{\sqrt{n}} \Rightarrow B_t.
\]

Eg 2:

- \( Q = \text{Bernoulli}(p) \), then \( Y_t \) is a simple random walk on the connected component of percolation

Eg 3:

- \( Q \) supported on \([1, \infty)\) – what we shall focus on
Two laws

- Two laws:
  1. **Quenched Law**: For any given realization $\omega$, study the law $P_\omega$ of $(X_t)/(Y_t)$ under this realization
  2. **Averaged (or Annealed) Law**: the law by taking expectation of the quenched law $P_\omega$ w.r.t. $\mathbb{P}$

- Focus on quenched law $P_\omega$
- Basic Questions: the long run behavior of $(X_t)/(Y_t)$, e.g.,
  1. does the quenched FCLT (QFCLT) hold?
  2. What about the fractal properties of the sample paths of $(X_t)/(Y_t)$?
QFCLT

- [Barlow and Deuschel(2010)] For the VSRW $X$, when $d \geq 2$, for $\mathbb{P}$-a.a. $\omega$, under $\mathbb{P}_0^\omega$, $X_{n^2 t}/n \Rightarrow \sigma_V B_t$, where $\sigma_V$ is non-random, and $B_t$ is a standard $d$-dimensional Brownian-motion.

- [Barlow and Deuschel(2010)] For the CSRW $Y$, when $d \geq 2$, for $\mathbb{P}$-a.a. $\omega$, under $\mathbb{P}_0^\omega$, $Y_{n^2 t}/n \Rightarrow \sigma_C B_t$,

where $\sigma_C = \begin{cases} \sigma_V / \sqrt{2d E\mu_e}, & \text{if } E\mu_e < \infty, \\ 0, & \text{if } E\mu_e = \infty. \end{cases}$

- [Barlow and Černý(2011)], [Černý(2011)] For the CSRW $Y$, when $d \geq 2$ and $Q(\mu_e \geq u) \sim C/u^\alpha$ for some $\alpha \in (0, 1)$, then for $\mathbb{P}$-a.a. $\omega$, under $\mathbb{P}_0^\omega$, $Y_{n^{2/\alpha} t}/n$ converges to a multiple of the fractional kinetics process;

- [Barlow and Zheng(2010)] For the CSRW $Y$, when $d \geq 3$ and $Q$ is Cauchy tailed, then for $\mathbb{P}$-a.a. $\omega$, under $\mathbb{P}_0^\omega$, $Y_{n^2(\log n)t}/n$ converges to a multiple of a $d$-dimensional Brownian-motion.
Discrete Hausdorff Dimension

- For any $n \in \mathbb{N}$, let $V_n = V(0, 2^n)$ be the cube of side length $2^n$ centered at $0 \in \mathbb{Z}^d$, and $S_n := V_n \setminus V_{n-1}$
- For any set $B \subseteq \mathbb{Z}^d$, let $s(B)$ be its side length
- [Barlow and Taylor (1992)] For any measure function $h$ and any set $A \subseteq \mathbb{Z}^d$, the discrete Hausdorff measure of $A$ w.r.t $h$ is
  \[ m_h(A) = \sum_{n=1}^{\infty} \nu_h(A, S_n). \]
  where
  \[ \nu_h(A, S_n) = \min \left\{ \sum_{i=1}^{k} h\left( \frac{s(B_i)}{2^n} \right) : A \cap S_n \subset \bigcup_{i=1}^{k} B_i \right\}. \]
- For $\alpha > 0$, define $h(r) = r^\alpha$, and let $m_\alpha(A) = m_h(A)$. Then the discrete Hausdorff dimension of $A$ is given by
  \[ \dim_H A = \inf \{ \alpha > 0 : m_\alpha(A) < \infty \}. \]
Discrete Packing Dimension

- [Barlow and Taylor(1992)] For any measure function $h$, $\varepsilon > 0$, and any set $A \subseteq \mathbb{Z}^d$, the discrete packing measure of $A$ w.r.t $h$ is

$$p_h(A, \varepsilon) = \sum_{n=1}^{\infty} \tau_h(A, S_n, \varepsilon),$$

where

$$\tau_h(A, S_n, \varepsilon) = \max \left\{ \sum_{i=1}^{k} h\left( \frac{r_i}{2^n} \right) : x_i \in A \cap S_n, V(x_i, r_i) \text{ disjoint}, 1 \leq r_i \leq 2^{(1-\varepsilon)n} \right\}$$

- Say that $A \subseteq \mathbb{Z}^d$ is $h$-packing finite if $p_h(A, \varepsilon) < \infty$ for all $\varepsilon \in (0, 1)$.

- The discrete packing dimension of $A$ is defined by

$$\dim_p A = \inf \left\{ \alpha > 0 : A \text{ is } r^\alpha\text{-packing finite} \right\}.$$
Discrete Dimensions of the Range of RCM

Theorem
[Xiao and Zheng (2011)] Let

\[ R = \{ x \in \mathbb{Z}^d : X_t = x \text{ for some } t \geq 0 \} \]

be the range of VSRW X (as well as that of CSRW Y). Assume that \( d \geq 3 \) and \( Q(\mu_e \geq 1) = 1 \). Then for \( P \)-almost every \( \omega \in \Omega \),

\[ \dim_H R = \dim_P R = 2, \quad P_0^\omega \text{-a.s.} \]

where \( \dim_H \) and \( \dim_P \) denote respectively the discrete Hausdorff and packing dimension.
Recurrent/Transient Sets for RCM

**Theorem**

[Xiao and Zheng(2011)] Assume that $d \geq 3$ and $\mathbb{P}(\mu_e \geq 1) = 1$. Let $A \subset \mathbb{Z}^d$ be any (infinite) set. Then for $\mathbb{P}$-almost every $\omega \in \Omega$, the following statements hold.

(i) If $\dim_H A < d - 2$, then

$$\mathbb{P}_0^\omega(X_t \in A \text{ for arbitrarily large } t > 0) = 0.$$  

(ii) If $\dim_H A > d - 2$, then

$$\mathbb{P}_0^\omega(X_t \in A \text{ for arbitrarily large } t > 0) = 1.$$  

**Remark**

Both theorems are also proven for the Bouchaud’s trap model.
Main Ingredients of Proof

- Basic idea: derive various estimates for ordinary random walks used in [Barlow and Taylor(1992)], by using general Markov chain techniques
- Main ingredients:
  1. Gaussian heat kernel bounds for the VSRW ([Barlow and Deuschel(2010)]);
  2. Hitting probability estimates;
  3. Tail probability estimates of the sojourn measure for the discrete time VSRW;
  4. Tail probability estimates of the maximal displacement of VSRW;
  5. A SLLN for dependent events;
  6. A zero-one law as a consequence of an elliptic Harnack inequality that the VSRW satisfies.
Proof Sketch for Theorem 1

- \( \dim_p R \leq 2 \) \( P_0^\omega \)-a.s.: first moment argument;
- \( \dim_H R \geq 2 \) \( P_0^\omega \)-a.s.: let \( \hat{R} \) be the range of the discrete time VSRW \( (\hat{Y}_n) := (Y_n) \), and show that \( \dim_H \hat{R} \geq 2 \).
  - Let \( \mu \) be the counting measure on \( \hat{R} \). Show that \( \mu(Q_k(x)) \leq cn2^{2k} \) for every \( x \in S_n \) and \( 0 \leq k \leq n \).
  - Frostman’s lemma \( \Rightarrow \)  
    \[ \nu_2(\hat{R}, S_n) \geq c^{-1} n^{-1} 2^{-2n} \mu(S_n) \]
  - Hitting probability estimate \( \Rightarrow \)  
    \[ E_0^\omega(\mu(S_n)) \geq c 2^{2n} \]
    and hence \( E_0^\omega(m_2(\hat{R})) = \infty \).
- To further prove \( m_2(\hat{R}) = \infty \) \( P_0^\omega \)-a.s., let \( n_k = \lceil \lambda k \log k \rceil \) for \( \lambda > 0 \) TBD, and define  
  \[ \tau_k = \inf \left\{ n > 0 : \hat{X}_n \notin \mathcal{V}(0, 2^{n_k}) \right\} \]
  Show that
  1. \( P_0^\omega\left(|\hat{X}_{\tau_k-1}| > 2^{n_k-3}\right) \leq c \exp(-ck) \); and
  2. On the event \( \{|\hat{X}_{\tau_k-1}| \leq 2^{n_k-3}\} \),  
    \( P_{\hat{X}_{\tau_k-1}}^\omega(\mu(S_{n_k}) \geq c 2^{2n_k}) \geq p \).
  3. The SLLN for dependent event concludes.
Summary

0. QFCLT for the VSRW/CSRW
1. Discrete fractal dimensions of the range of VSRW/CSRW
2. Characterization of recurrent/transient sets for VSRW/CSRW
3. Similarly for Bouchaud’s trap model.

Thank you!


